

SHORT FORMULAS FOR ALGEBRAIC COVARIANT DERIVATIVE CURVATURE TENSORS VIA ALGEBRAIC COMBINATORICS

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ABSTRACT. We consider generators of algebraic covariant derivative curvature tensors \mathfrak{R}' which can be constructed by a Young symmetrization of product tensors $W \otimes U$ or $U \otimes W$, where W and U are covariant tensors of order 2 and 3. W is a symmetric or alternating tensor whereas U belongs to a class of the infinite set \mathfrak{S} of irreducible symmetry classes characterized by the partition (21). Using Computer Algebra we search for such generators whose coordinate representations are polynomials with a minimal number of summands. For a generic choice of the symmetry class of U we obtain lengths of 16 or 20 summands if W is symmetric or skew-symmetric, respectively. In special cases these numbers can be reduced to the minima 12 or 10. If these minima occur then U admits an index commutation symmetry. Furthermore minimal lengths are possible if U is formed from torsion-free covariant derivatives of symmetric or alternating 2-tensor fields.

Foundation of our investigations is a theorem of S. A. Fulling, R. C. King, B. G. Wybourne and C. J. Cummins about a Young symmetrizer that generates the symmetry class of algebraic covariant derivative curvature tensors. Furthermore we apply ideals and idempotents in group rings $\mathbb{C}[\mathcal{S}_r]$ and discrete Fourier transforms for symmetric groups \mathcal{S}_r . For symbolic calculations we used the Mathematica packages Ricci and PERMS.

1. INTRODUCTION

The present paper continues investigations of [13, 14] in which we constructed generators of *algebraic curvature tensors* and *algebraic covariant derivative curvature tensors*.

Algebraic curvature tensors are tensors of order 4 which have the same symmetry properties as the Riemann tensor of a Levi-Civita connection in Differential Geometry. Let $\mathcal{T}_r V$ be the vector space of the r -times covariant tensors T over a finite-dimensional \mathbb{K} -vector space V , $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. We assume that V possesses a *fundamental tensor* $g \in \mathcal{T}_2 V$ (of arbitrary signature) which can be used for raising and lowering of tensor indices.

Definition 1.1. A tensor $\mathfrak{R} \in \mathcal{T}_4 V$ is called an *algebraic curvature tensor* iff \mathfrak{R} has the *index commutation symmetry*

$$(1.1) \quad \forall w, x, y, z \in V : \quad \mathfrak{R}(w, x, y, z) = -\mathfrak{R}(w, x, z, y) = \mathfrak{R}(y, z, w, x)$$

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and fulfills the *first Bianchi identity*

$$(1.2) \quad \forall w, x, y, z \in V : \quad \mathfrak{R}(w, x, y, z) + \mathfrak{R}(w, y, z, x) + \mathfrak{R}(w, z, x, y) = 0.$$

Definition 1.2. A tensor $\mathfrak{R}' \in \mathcal{T}_5 V$ is called an *algebraic covariant derivative curvature tensor* iff \mathfrak{R}' has the *index commutation symmetry*

$$(1.3) \quad \mathfrak{R}'(w, x, y, z, u) = -\mathfrak{R}'(w, x, z, y, u) = \mathfrak{R}'(y, z, w, x, u)$$

and fulfills the *first Bianchi identity*

$$(1.4) \quad \mathfrak{R}'(w, x, y, z, u) + \mathfrak{R}'(w, y, z, x, u) + \mathfrak{R}'(w, z, x, y, u) = 0$$

and the *second Bianchi identity*

$$(1.5) \quad \mathfrak{R}'(w, x, y, z, u) + \mathfrak{R}'(w, x, z, u, y) + \mathfrak{R}'(w, x, u, y, z) = 0$$

for all $u, w, x, y, z \in V$.

The relations (1.1) – (1.5) correspond to the well-known formulas

$$(1.6) \quad R_{ijkl} = -R_{ijlk} = R_{klij}$$

$$(1.7) \quad R_{ijkl} + R_{iklj} + R_{iljk} = 0$$

for the Riemann tensor R and

$$(1.8) \quad R_{ijkl;m} = -R_{ijlk;m} = R_{klij;m}$$

$$(1.9) \quad R_{ijkl;m} + R_{iklj;m} + R_{iljk;m} = 0$$

$$(1.10) \quad R_{ijkl;m} + R_{ijlm;k} + R_{ijmk;l} = 0$$

for its first covariant derivative which we present here in terms of tensor coordinates with respect to an arbitrary local coordinate system.

A famous problem connected with algebraic curvature tensors is the *Osserman conjecture*.

Definition 1.3. Let $\mathfrak{R} \in \mathcal{T}_4 V$ be an algebraic curvature tensor. For $x \in V$, the *Jacobi operator* $J_{\mathfrak{R}}(x)$ of \mathfrak{R} and x is the linear operator $J_{\mathfrak{R}}(x) : V \rightarrow V$, $J_{\mathfrak{R}}(x) : y \mapsto J_{\mathfrak{R}}(x)y$ that is defined by $\forall w \in V : g(J_{\mathfrak{R}}(x)y, w) = \mathfrak{R}(y, x, x, w)$.

Definition 1.4. An algebraic curvature tensor \mathfrak{R} is called *Osserman* if the eigenvalues of $J_{\mathfrak{R}}(x)$ are constant both on $S^+(V) := \{x \in V \mid g(x, x) = +1\}$ and on $S^-(V) := \{x \in V \mid g(x, x) = -1\}$.

If R is the Riemann tensor of a Riemannian manifold (M, g) which is locally a rank one symmetric space or flat, then the eigenvalues of $J_R(x)$ are constant on the unit sphere bundle of (M, g) . Osserman [33] wondered if the converse held. This question is known as the *Osserman conjecture*.

The correctness of the Osserman conjecture has been established for Riemannian manifolds (M, g) in all dimensions $\neq 8, 16$ (see [4, 32]) and for Lorentzian manifolds (M, g) in all dimensions (see [1, 19]). However Osserman's question has a negative answer in the case of a pseudo-Riemannian metric with signature (p, q) , $p, q \geq 2$ (see e.g. the references given in [13, p.2]). A detailed view about the Osserman conjecture can be found in the book [21] by P. B. Gilkey.

Numerous examples of Osserman algebraic curvature tensors can be constructed by means of operators α and γ given below. It turned out that these operators lead to generators for arbitrary algebraic curvature tensors.

Definition 1.5. Let $S, A \in \mathcal{T}_2V$ be symmetric or alternating tensors of order 2, i.e. their coordinates satisfy $S_{ij} = S_{ji}$, $A_{ij} = -A_{ji}$. We define tensors $\gamma(S), \alpha(A) \in \mathcal{T}_4V$ by

$$(1.11) \quad \gamma(S)_{ijkl} := \frac{1}{3} (S_{il}S_{jk} - S_{ik}S_{jl}) ,$$

$$(1.12) \quad \alpha(A)_{ijkl} := \frac{1}{3} (2 A_{ij}A_{kl} + A_{ik}A_{jl} - A_{il}A_{jk}) .$$

Now we can construct an example of an Osserman algebraic curvature tensor in the following way. Let $g \in \mathcal{T}_2V$ be a positive definite metric and $\{C_i\}_{i=1}^r$ be a finite set of real, skew-symmetric $(\dim V \times \dim V)$ -matrices that satisfy the Clifford commutation relations $C_i \cdot C_j + C_j \cdot C_i = -2\delta_{ij} \text{Id}$. If we form skew-symmetric tensors $A_i \in \mathcal{T}_2V$ by $A_i(x, y) := g(C_i \cdot x, y)$ ($x, y \in V$), then

$$(1.13) \quad \mathfrak{R} = \lambda_0 \gamma(g) + \sum_{i=1}^r \lambda_i \alpha(A_i) \quad , \quad \lambda_0, \lambda_i = \text{const.}$$

is an Osserman algebraic curvature tensor (see [20]). Further examples which allow also indefinite metrics can be found in [21, pp.191-193]. (See also [13, Sec.6].)

The operators α and γ can be used to form generators for arbitrary algebraic curvature tensors. P. Gilkey [21, pp.41-44] and B. Fiedler [13] gave different proofs for

Theorem 1.6. *Each of the sets of tensors*

- (1) $\{\gamma(S) \mid S \in \mathcal{T}_2V \text{ symmetric}\}$
- (2) $\{\alpha(A) \mid A \in \mathcal{T}_2V \text{ skew-symmetric}\}$

generates the vector space of all algebraic curvature tensors \mathfrak{R} on V .

Note that the tensors $\gamma(S)$ and $\alpha(A)$ are expressions which arise from $S \otimes S$ or $A \otimes A$ by a symmetrization $\gamma(S) = \frac{1}{12} y_t^*(S \otimes S)$, $\alpha(A) = \frac{1}{12} y_t^*(A \otimes A)$, where y_t is the Young symmetrizer of the Young tableau

$$(1.14) \quad t = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} .$$

(See [13]. See also Section 2 for basic facts and definitions.)

In [14] we searched for similar *generators of algebraic covariant derivative curvature tensors*. We used Boerner's definition of *symmetry classes* for tensors $T \in \mathcal{T}_rV$ by right ideals $\mathfrak{r} \subseteq \mathbb{K}[\mathcal{S}_r]$ of the group ring $\mathbb{K}[\mathcal{S}_r]$ of the symmetric group \mathcal{S}_r (see Section 2 and [2, 3, 9, 12]). On this basis we investigated the following

Problem 1.7. We search for generators of algebraic covariant derivative curvature tensors which can be formed by a suitable symmetry operator from tensors

$$(1.15) \quad T \otimes \hat{T} \quad \text{or} \quad \hat{T} \otimes T \quad , \quad T \in \mathcal{T}_2V \quad , \quad \hat{T} \in \mathcal{T}_3V$$

where T and \hat{T} belongs to symmetry classes of \mathcal{T}_2V and \mathcal{T}_3V which are defined by minimal right ideals $\mathfrak{r} \subset \mathbb{K}[\mathcal{S}_2]$ and $\hat{\mathfrak{r}} \subset \mathbb{K}[\mathcal{S}_3]$, respectively.

All such generators can be gained by means of the *Young symmetrizer* y_ν of the *Young tableau*

$$(1.16) \quad t' = \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 & \\ \hline \end{array} .$$

In [14] we proved the following Theorems 1.8–1.10:

Theorem 1.8. *A solution of Problem 1.7 can be constructed at most from such pairs of tensors (1.15) whose symmetry classes are characterized by the following partitions $\lambda \vdash 2$, $\hat{\lambda} \vdash 3$:*

	λ for T	$\hat{\lambda}$ for \hat{T}
(a)	(2), i.e. T symmetric	(3), i.e. \hat{T} symmetric
(b)	(2), i.e. T symmetric	(2 1)
(c)	(1 ²), i.e. T skew-symmetric	(2 1)

The proof of Theorem 1.8 is based on the *Littlewood-Richardson rule*¹. (see [14]). The case (a) of Theorem 1.8 is specified by

Theorem 1.9. *Let us denote by $S \in \mathcal{T}_2 V$ and $\hat{S} \in \mathcal{T}_3 V$ symmetric tensors of order 2 and 3, respectively. Then the set of all tensors which belong to exactly one of the following tensor types*

$$(1.17) \quad \tau : \quad y_\nu^*(S \otimes \hat{S}) \quad , \quad y_\nu^*(\hat{S} \otimes S) \quad ,$$

generates the vector space of all algebraic covariant derivative curvature tensors $\mathfrak{R}' \in \mathcal{T}_5 V$. Moreover, the tensors (1.17) coincide and their coordinates fulfill

$$(1.18) \quad \begin{aligned} \hat{\gamma}(S, \hat{S})_{ijkl} &:= (y_\nu^*(S \otimes \hat{S}))_{ijkl} = (y_\nu^*(\hat{S} \otimes S))_{ijkl} \\ &= 4 \left\{ S_{il} \hat{S}_{jks} - S_{jl} \hat{S}_{iks} + S_{jk} \hat{S}_{ils} - S_{ik} \hat{S}_{jls} \right\} \end{aligned}$$

The operator $\hat{\gamma}$ plays the same role for the generators of algebraic covariant derivative curvature tensors considered in Theorem 1.9 as the operators α and γ play for the generators of algebraic curvature tensors. A first proof that the expressions (1.18) are generators for \mathfrak{R}' was given by P. B. Gilkey [21, p.236].

The cases (b) and (c) of Theorem 1.8 lead to

Theorem 1.10. *Let us denote by $S, A \in \mathcal{T}_2 V$ symmetric or alternating tensors of order 2 and by $U \in \mathcal{T}_3 V$ covariant tensors of order 3 whose symmetry class $\mathcal{T}_\mathfrak{r}$ is defined by a fixed minimal right ideal $\mathfrak{r} \subset \mathbb{K}[\mathcal{S}_3]$ from the equivalence class characterized by the partition $(2, 1) \vdash 3$. We consider the following types τ of tensors*

$$(1.19) \quad \tau : \quad y_\nu^*(S \otimes U) \quad , \quad y_\nu^*(U \otimes S) \quad , \quad y_\nu^*(A \otimes U) \quad , \quad y_\nu^*(U \otimes A) .$$

Then for each of the above types τ the following assertions are equivalent:

¹See the references [23, 24, 22, 28, 29, 16, 17] for the Littlewood-Richardson rule.

- (1) *The vector space of algebraic covariant derivative curvature tensors $\mathfrak{R}' \in \mathcal{T}_5 V$ is the set of all finite sums of tensors of the type τ considered.*
- (2) *The right ideal \mathfrak{r} is different from the right ideal $\mathfrak{r}_0 := f \cdot \mathbb{K}[\mathcal{S}_3]$ with generating idempotent*

$$(1.20) \quad f := \left\{ \frac{1}{2}(\text{id} - (1\ 3)) - \frac{1}{6}y \right\} \quad , \quad y := \sum_{p \in \mathcal{S}_3} \text{sign}(p) p.$$

In Theorem 1.10 not only a single symmetry class is allowed for the tensors U but the *complete infinite set* \mathfrak{S} of irreducible symmetry classes of $(2\ 1) \vdash 3$ from which only the class of the right ideal \mathfrak{r}_0 has to be excluded.

In the situation of Theorem 1.10 we can also determine operators of the type α , γ , $\hat{\gamma}$ which describe the generators of the algebraic covariant derivative curvature tensors \mathfrak{R}' considered. However, these operators depend on the right ideal \mathfrak{r} (or its generating idempotent e) that defines the symmetry class of U . And they yield no short expressions of 2, 3, or 4 terms but longer expressions between 10 and 20 terms of length. *The search for shortest expressions of this type is the subject of our paper.* Some of our main results are collected in

Theorem 1.11. *Consider the situation of Theorem 1.10. Then it holds:*

- (1) *The tensors (1.19) satisfy*

$$(1.21) \quad y_\nu^*(S \otimes U) = y_\nu^*(U \otimes S) \quad , \quad y_\nu^*(A \otimes U) = -y_\nu^*(U \otimes A).$$

- (2) *Let $\dim V \geq 3$. Then the coordinates of (1.19) are sums of the following lengths*

		$y_\nu^*(S \otimes U)$	$y_\nu^*(A \otimes U)$
(a)	generic case for \mathfrak{r}	16	20
(b)	\mathfrak{r} producing minimal lengths	12	10

- (3) *For $\dim V \geq 3$ there exist minimal right ideals \mathfrak{r} of $(2\ 1) \vdash 3$ which lead to the minimal lengths of case (b) both for $y_\nu^*(S \otimes U)$ and for $y_\nu^*(A \otimes U)$.*
- (4) *If the coordinates of $y_\nu^*(S \otimes U)$ or $y_\nu^*(A \otimes U)$ have the minimal lengths of case (b) and $\dim V \geq 3$ then U admits an index commutation symmetry.*

Further results are given in Section 4 and the Appendices of our paper.

The concept "expression of minimal length" depends on the method which we use to reduce expressions (see Section 3). Remark 3.9 discusses a generalization of our reduction method which could possibly lead to a further decrease of the numbers in Theorem 1.11.

Examples of tensors U with a symmetry from \mathfrak{S} are given by tensor fields

$$(1.22) \quad U = (\nabla\psi - \text{sym}(\nabla\psi))|_p \quad \text{or} \quad U = (\nabla\omega - \text{alt}(\nabla\omega))|_p \quad , \quad p \in M,$$

where $\psi, \omega \in \mathcal{T}_2 M$ are symmetric/alternating tensor fields of order 2 on a differentiable manifold M and ∇ is a torsion-free covariant derivative (see [15] and Section 5). For tensors (1.22) we obtained the following result

Theorem 1.12. *If we consider tensors U, S, A on a tangent space $V = T_p M$ of a differentiable manifold M , $\dim M \geq 3$, and generate U by one of the formulas*

(1.22) then we obtain the shortest lengths from Theorem 1.11, (2b) exactly in the following cases:

- (1) $y_{\nu}^*(U \otimes S)$ and $U = (\nabla\psi - \text{sym}(\nabla\psi))|_p$, $\psi \in \mathcal{T}_2M$ symmetric,
- (2) $y_{\nu}^*(U \otimes S)$, $y_{\nu}^*(U \otimes A)$ and $U = (\nabla\omega - \text{alt}(\nabla\omega))|_p$, $\omega \in \mathcal{T}_2M$ skew-symmetric.

Here is a brief outline to the paper. In Section 2 we give a summary of basic facts about symmetry classes, Young symmetrizers and discrete Fourier transforms. These tools are needed to obtain the infinite set \mathfrak{S} of symmetry classes for U . In Section 3 and 4 we construct short coordinate representations for the tensors (1.19) by determining and solving a complete system of linear identities for the tensors U . In Section 5 we determine the conditions for the occurrence of index commutation symmetries on the tensors U and show statement (4) of Theorem 1.11. In Section 6 we prove Theorem 1.12. Many results were obtained by computer calculations by means of the **Mathematica** packages **Ricci** [27] and **PERMS** [10]. The **Mathematica** notebooks of these calculations are available at [7].

2. BASIC FACTS

The vector spaces of algebraic curvature tensors or algebraic covariant derivative tensors over V are *symmetry classes* in the sence of H. Boerner [2, p.127]. We denote by $\mathbb{K}[\mathcal{S}_r]$ the *group ring* of a symmetric group \mathcal{S}_r over the field \mathbb{K} . Every group ring element $a = \sum_{p \in \mathcal{S}_r} a(p) p \in \mathbb{K}[\mathcal{S}_r]$ acts as so-called *symmetry operator* on tensors $T \in \mathcal{T}_r V$ according to the definition

$$(2.1) \quad (aT)(v_1, \dots, v_r) := \sum_{p \in \mathcal{S}_r} a(p) T(v_{p(1)}, \dots, v_{p(r)}) \quad , \quad v_i \in V .$$

Equation (2.1) is equivalent to $(aT)_{i_1 \dots i_r} = \sum_{p \in \mathcal{S}_r} a(p) T_{i_{p(1)} \dots i_{p(r)}}$.

Definition 2.1. Let $\mathfrak{r} \subseteq \mathbb{K}[\mathcal{S}_r]$ be a right ideal of $\mathbb{K}[\mathcal{S}_r]$ for which an $a \in \mathfrak{r}$ and a $T \in \mathcal{T}_r V$ exist such that $aT \neq 0$. Then the tensor set

$$(2.2) \quad \mathcal{T}_{\mathfrak{r}} := \{aT \mid a \in \mathfrak{r}, T \in \mathcal{T}_r V\}$$

is called the *symmetry class* of tensors defined by \mathfrak{r} .

Since $\mathbb{K}[\mathcal{S}_r]$ is semisimple for $\mathbb{K} = \mathbb{R}, \mathbb{C}$, every right ideal $\mathfrak{r} \subseteq \mathbb{K}[\mathcal{S}_r]$ possesses a generating idempotent e , i.e. \mathfrak{r} fulfils $\mathfrak{r} = e \cdot \mathbb{K}[\mathcal{S}_r]$. It holds (see e.g. [13] or [2, 3])

Lemma 2.2. *If e is a generating idempotent of \mathfrak{r} , then a tensor $T \in \mathcal{T}_r V$ belongs to $\mathcal{T}_{\mathfrak{r}}$ iff $eT = T$. Thus we have $\mathcal{T}_{\mathfrak{r}} = \{eT \mid T \in \mathcal{T}_r V\}$.*

Now we summarize tools from our Habilitationsschrift [11] (see also its summary [12]). We make use of the following connection between r -times covariant tensors $T \in \mathcal{T}_r V$ and elements of the *group ring* $\mathbb{K}[\mathcal{S}_r]$.

Definition 2.3. Any tensor $T \in \mathcal{T}_r V$ and any r -tuple $b := (v_1, \dots, v_r) \in V^r$ of r vectors from V induce a function $T_b : \mathcal{S}_r \rightarrow \mathbb{K}$ according to the rule

$$(2.3) \quad T_b(p) := T(v_{p(1)}, \dots, v_{p(r)}) \quad , \quad p \in \mathcal{S}_r .$$

We identify this function with the group ring element $T_b := \sum_{p \in \mathcal{S}_r} T_b(p) p \in \mathbb{K}[\mathcal{S}_r]$.

Obviously, two tensors $S, T \in \mathcal{T}_r V$ fulfil $S = T$ iff $S_b = T_b$ for all $b \in V^r$. We denote by $'*$ ' the mapping $*$: $a = \sum_{p \in \mathcal{S}_r} a(p) p \mapsto a^* := \sum_{p \in \mathcal{S}_r} a(p) p^{-1}$. Then the following important formula² holds

$$(2.4) \quad \forall T \in \mathcal{T}_r V, a \in \mathbb{K}[\mathcal{S}_r], b \in V^r : (aT)_b = T_b \cdot a^*.$$

Now it can be shown that all T_b of tensors T of a given symmetry class lie in a certain left ideal of $\mathbb{K}[\mathcal{S}_r]$.

Proposition 2.4.³ *Let $e \in \mathbb{K}[\mathcal{S}_r]$ be an idempotent. Then a $T \in \mathcal{T}_r V$ fulfils the condition $eT = T$ iff $T_b \in \mathfrak{l} := \mathbb{K}[\mathcal{S}_r] \cdot e^*$ for all $b \in V^r$, i.e. all T_b of T lie in the left ideal \mathfrak{l} generated by e^* .*

The proof follows easily from (2.4). Since a right ideal \mathfrak{r} defining a symmetry class and the left ideal \mathfrak{l} from Proposition 2.4 satisfy $\mathfrak{r} = \mathfrak{l}^*$, we denote symmetry classes also by $\mathcal{T}_{\mathfrak{r}^*}$. A further result is

Proposition 2.5.⁴ *If $\dim V \geq r$, then every left ideal $\mathfrak{l} \subseteq \mathbb{K}[\mathcal{S}_r]$ fulfils $\mathfrak{l} = \mathcal{L}_{\mathbb{K}}\{T_b \mid T \in \mathcal{T}_{\mathfrak{r}^*}, b \in V^r\}$. (Here $\mathcal{L}_{\mathbb{K}}$ denotes the forming of the linear closure.)*

If $\dim V < r$, then the T_b of the tensors from $\mathcal{T}_{\mathfrak{r}^*}$ will span only a linear subspace of \mathfrak{l} in general.

Important special symmetry operators are Young symmetrizers, which are defined by means of Young tableaux.

A *Young tableau* t of $r \in \mathbb{N}$ is an arrangement of r boxes such that

- (1) the numbers λ_i of boxes in the rows $i = 1, \dots, l$ form a decreasing sequence $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_l > 0$ with $\lambda_1 + \dots + \lambda_l = r$,
- (2) the boxes are fulfilled by the numbers $1, 2, \dots, r$ in any order.

For instance, the following graphics shows a Young tableau of $r = 16$.

$$\left. \begin{array}{l} \lambda_1 = 5 \\ \lambda_2 = 4 \\ \lambda_3 = 4 \\ \lambda_4 = 2 \\ \lambda_5 = 1 \end{array} \right\} \begin{array}{|c|c|c|c|c|} \hline 11 & 2 & 5 & 4 & 12 \\ \hline 9 & 6 & 16 & 15 & \\ \hline 8 & 14 & 1 & 7 & \\ \hline 13 & 3 & & & \\ \hline 10 & & & & \\ \hline \end{array} \right\} = t.$$

Obviously, the unfilled arrangement of boxes, the *Young frame*, is characterized by a partition $\lambda = (\lambda_1, \dots, \lambda_l) \vdash r$ of r .

If a Young tableau t of a partition $\lambda \vdash r$ is given, then the *Young symmetrizer* y_t of t is defined by⁵

$$(2.5) \quad y_t := \sum_{p \in \mathcal{H}_t} \sum_{q \in \mathcal{V}_t} \text{sign}(q) p \circ q$$

²See B. Fiedler [9, Sec.III.1] and B. Fiedler [11].

³See B. Fiedler [11] or B. Fiedler [9, Prop. III.2.5, III.3.1, III.3.4].

⁴See B. Fiedler [11] or B. Fiedler [9, Prop. III.2.6].

⁵We use the convention $(p \circ q)(i) := p(q(i))$ for the product of two permutations p, q .

where $\mathcal{H}_t, \mathcal{V}_t$ are the groups of the *horizontal* or *vertical permutations* of t which only permute numbers within rows or columns of t , respectively. The Young symmetrizers of $\mathbb{K}[\mathcal{S}_r]$ are essentially idempotent and define decompositions

$$(2.6) \quad \mathbb{K}[\mathcal{S}_r] = \bigoplus_{\lambda \vdash r} \bigoplus_{t \in \mathcal{ST}_\lambda} \mathbb{K}[\mathcal{S}_r] \cdot y_t \quad , \quad \mathbb{K}[\mathcal{S}_r] = \bigoplus_{\lambda \vdash r} \bigoplus_{t \in \mathcal{ST}_\lambda} y_t \cdot \mathbb{K}[\mathcal{S}_r]$$

of $\mathbb{K}[\mathcal{S}_r]$ into minimal left or right ideals. In (2.6), the symbol \mathcal{ST}_λ denotes the set of all standard tableaux of the partition λ . Standard tableaux are Young tableaux in which the entries of every row and every column form an increasing number sequence.⁶

S.A. Fulling, R.C. King, B.G. Wybourne and C.J. Cummins showed in [16] that the symmetry classes of the Riemannian curvature tensor R and its *symmetrized⁷ covariant derivatives*

$$(2.7) \quad (\nabla^{(u)} R)_{ijkl s_1 \dots s_u} := \nabla_{(s_1} \nabla_{s_2} \dots \nabla_{s_u)} R_{ijkl} = R_{ijkl; (s_1 \dots s_u)}$$

are generated by special Young symmetrizers.

Theorem 2.6. *Consider the Levi-Civita connection ∇ of a pseudo-Riemannian metric g . For $u \geq 0$ the Riemann tensor and its symmetrized covariant derivatives $\nabla^{(u)} R$ fulfil*

$$(2.8) \quad e_t^* \nabla^{(u)} R = \nabla^{(u)} R$$

where $e_t := y_t(u+1)/(2 \cdot (u+3)!)$ is an idempotent which is formed from the Young symmetrizer y_t of the standard tableau

$$(2.9) \quad t = \begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 5 & \dots & \dots \\ \hline 2 & 4 & & & (u+4) \\ \hline \end{array}.$$

A proof of this result of [16] can be found in [8, Sec.6], too. The proof needs only the symmetry properties (1.1) or (1.6), the identities Bianchi I and Bianchi II and the symmetry with respect to s_1, \dots, s_u . Thus Theorem 2.6 is a statement about algebraic curvature tensors and algebraic covariant derivative curvature tensors. We can specify this in the following way:

Definition 2.7. A tensor $\mathfrak{R}^{(u)} \in \mathcal{T}_{4+u}V$, $u \geq 0$, is called a *symmetric algebraic covariant derivative curvature tensor of order u* iff $\mathfrak{R}^{(u)}(w, y, z, x, a_1, \dots, a_u)$ is symmetric with respect to a_1, \dots, a_u and fulfills

$$(2.10) \quad \begin{aligned} \mathfrak{R}^{(u)}(w, x, y, z, a_1, \dots, a_u) &= -\mathfrak{R}^{(u)}(w, x, z, y, a_1, \dots, a_u) \\ &= \mathfrak{R}^{(u)}(y, z, w, x, a_1, \dots, a_u) \end{aligned}$$

⁶About Young symmetrizers and Young tableaux see for instance [2, 3, 16, 18, 22, 23, 28, 29, 30, 31, 34, 35]. In particular, properties of Young symmetrizers in the case $\mathbb{K} \neq \mathbb{C}$ are described in [30].

⁷(...) denotes the symmetrization with respect to the indices s_1, \dots, s_u .

$$(2.11) \quad 0 = \mathfrak{R}^{(u)}(w, x, y, z, a_1, \dots, a_u) + \mathfrak{R}^{(u)}(w, y, z, x, a_1, \dots, a_u) + \mathfrak{R}^{(u)}(w, z, x, y, a_1, \dots, a_u)$$

$$(2.12) \quad 0 = \mathfrak{R}^{(u)}(w, x, y, z, a_1, a_2, \dots, a_u) + \mathfrak{R}^{(u)}(w, x, z, a_1, y, a_2, \dots, a_u) + \mathfrak{R}^{(u)}(w, x, a_1, y, z, a_2, \dots, a_u)$$

for all $a_1, \dots, a_u, w, x, y, z \in V$.

Now symmetric algebraic covariant derivative curvature tensors can be characterized by means of the Young symmetrizer of the tableau (2.9).

Proposition 2.8. *A tensor $T \in \mathcal{T}_{4+u}V$, $u \geq 0$, is a symmetric algebraic covariant derivative curvature tensor of order u iff T satisfies*

$$(2.13) \quad e_t^* T = T$$

where e_t is the idempotent from Theorem 2.6.

A proof of Proposition 2.8 is given in the proof of [8, Prop.6.1]. If we consider now the values $u = 0$ and $u = 1$, we obtain

Corollary 2.9. *A tensor $T \in \mathcal{T}_4V$ [$\tilde{T} \in \mathcal{T}_5$] is an algebraic [covariant derivative] curvature tensor iff T [\tilde{T}] satisfies*

$$(2.14) \quad y_t^* T = 12 T \quad , \quad \left[y_{t'}^* \tilde{T} = 24 \tilde{T} \right]$$

where y_t [$y_{t'}$] is the Young symmetrizer of the standard tableau

$$(2.15) \quad t = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \quad , \quad \left[t' = \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 & \\ \hline \end{array} \right] .$$

In the situation considered in Theorem 1.10 the symmetry class of the tensors U is not unique. The (2 1)-equivalence class of minimal right ideals $\mathfrak{r} \subset \mathbb{K}[\mathcal{S}_3]$ which we use to define symmetry classes for the U is an infinite set. The set of generating idempotents for these right ideals \mathfrak{r} is infinite, too. In [14] we used *discrete Fourier transforms* to determine a family of primitive generating idempotents of the above minimal right ideals $\mathfrak{r} \subset \mathbb{K}[\mathcal{S}_3]$.

We denote by $\mathbb{K}^{d \times d}$ the set of all $d \times d$ -matrices of elements of \mathbb{K} .

Definition 2.10. A *discrete Fourier transform*⁸ for \mathcal{S}_r is an isomorphism

$$(2.16) \quad D : \mathbb{K}[\mathcal{S}_r] \rightarrow \bigotimes_{\lambda \vdash r} \mathbb{K}^{d_\lambda \times d_\lambda}$$

according to Wedderburn's theorem which maps the group ring $\mathbb{K}[\mathcal{S}_r]$ onto an outer direct product $\bigotimes_{\lambda \vdash r} \mathbb{K}^{d_\lambda \times d_\lambda}$ of full matrix rings $\mathbb{K}^{d_\lambda \times d_\lambda}$. We denote by D_λ the *natural projections* $D_\lambda : \mathbb{K}[\mathcal{S}_r] \rightarrow \mathbb{K}^{d_\lambda \times d_\lambda}$.

⁸See M. Clausen and U. Baum [5, 6] for details about fast discrete Fourier transforms.

A discrete Fourier transform maps every group ring element $a \in \mathbb{K}[\mathcal{S}_r]$ to a block diagonal matrix

$$(2.17) \quad D : a = \sum_{p \in \mathcal{S}_r} a(p) p \mapsto \begin{pmatrix} A_{\lambda_1} & & 0 \\ & A_{\lambda_2} & \\ & & \ddots \\ 0 & & & A_{\lambda_k} \end{pmatrix}.$$

The matrices A_λ are numbered by the partitions $\lambda \vdash r$. The dimension d_λ of every matrix $A_\lambda \in \mathbb{K}^{d_\lambda \times d_\lambda}$ can be calculated from the Young frame belonging to $\lambda \vdash r$ by means of the *hook length formula*. For $r = 3$ we have

$$\frac{\lambda \mid (3) \quad (21) \quad (1^3)}{d_\lambda \mid 1 \quad 2 \quad 1}$$

The inverse discrete Fourier transform is given by

Proposition 2.11. ⁹ *If $D : \mathbb{K}[\mathcal{S}_r] \rightarrow \bigotimes_{\lambda \vdash r} \mathbb{K}^{d_\lambda \times d_\lambda}$ is a discrete Fourier transform for $\mathbb{K}[\mathcal{S}_r]$, then we have for every $a \in \mathbb{K}[\mathcal{S}_r]$*

$$(2.18) \quad \begin{aligned} \forall p \in \mathcal{S}_r : \quad a(p) &= \frac{1}{r!} \sum_{\lambda \vdash r} d_\lambda \operatorname{trace} \{ D_\lambda(p^{-1}) \cdot D_\lambda(a) \} \\ &= \frac{1}{r!} \sum_{\lambda \vdash r} d_\lambda \operatorname{trace} \{ D_\lambda(p^{-1}) \cdot A_\lambda \}. \end{aligned}$$

In our considerations we are interested in the matrix ring $\mathbb{K}^{2 \times 2}$ which corresponds to the (21) -equivalence class of minimal right ideals $\mathfrak{r} \subset \mathbb{K}[\mathcal{S}_3]$. In [14] we proved

Proposition 2.12. *Every minimal right ideal $\mathfrak{r} \subset \mathbb{K}^{2 \times 2}$ is generated by exactly one of the following (primitive) idempotents*

$$(2.19) \quad Y := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad X_\nu := \begin{pmatrix} 1 & 0 \\ \nu & 0 \end{pmatrix}, \quad \nu \in \mathbb{K}.$$

Using an inverse discrete Fourier transform we can determine the primitive idempotents $\eta, \xi_\nu \in \mathbb{K}[\mathcal{S}_3]$ which correspond to Y, X_ν in (2.19). We calculated these idempotents by means of the tool `InvFourierTransform` of the Mathematica package `PERMS` [10] (see also [9, Appendix B].) This calculation can be carried out also by the program package `SYMMETRICA` [25, 26].

Proposition 2.13. *Let us use Young's natural representation¹⁰ of \mathcal{S}_3 as discrete Fourier transform. If we apply the Fourier inversion formula (2.18) to a 4×4 -block*

⁹See M. Clausen and U. Baum [5, p.81].

¹⁰Three discrete Fourier transforms (2.16) are known for symmetric groups \mathcal{S}_r : (1) *Young's natural representation*, (2) *Young's seminormal representation* and (3) *Young's orthogonal representation*. See [2, 3, 23, 5]. A short description of (1) and (2) can be found in [9, Sec.I.5.2]. All three discrete Fourier transforms are implemented in the program package `SYMMETRICA` [25, 26]. `PERMS` [10] uses the natural representation. The fast DFT-algorithm of M. Clausen and U. Baum [5, 6] is based on the seminormal representation.

matrix

$$(2.20) \quad \begin{pmatrix} A_{(3)} & & 0 \\ & A_{(21)} & \\ 0 & & A_{(1^3)} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where A is equal to X_ν or Y in (2.19), then we obtain the following idempotents of $\mathbb{K}[\mathcal{S}_3]$:

$$(2.21) \quad X_\nu \Rightarrow \xi_\nu := \frac{1}{3} \{ [1, 2, 3] + \nu[1, 3, 2] + (1 - \nu)[2, 1, 3] - \nu[2, 3, 1] + (-1 + \nu)[3, 1, 2] - [3, 2, 1] \}$$

$$(2.22) \quad Y \Rightarrow \eta := \frac{1}{3} \{ [1, 2, 3] - [2, 1, 3] - [2, 3, 1] + [3, 2, 1] \}.$$

Remark 2.14. It is interesting to clear up the connection of the idempotents ξ_ν and η with Young symmetrizers. A simple calculation shows that

$$(2.23) \quad \xi_0 = \frac{1}{3} y_{t_1} \quad , \quad \eta = \frac{1}{3} y_{t_2}$$

where y_{t_1} and y_{t_2} are the Young symmetrizers of the tableaux

$$t_1 = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array} \quad , \quad t_2 = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array} .$$

Remark 2.15. The proof of Theorem 1.10 is based on the following idea. To treat expressions $y_{t'}^*(S \otimes U)$ and $y_{t'}^*(A \otimes U)$ we form the following elements of $\mathbb{K}[\mathcal{S}_5]$:

$$(2.24) \quad \sigma_{\nu, \epsilon} := y_{t'}^* \cdot \zeta'_\epsilon \cdot \xi''_\nu \quad , \quad \rho_\epsilon := y_{t'}^* \cdot \zeta'_\epsilon \cdot \eta''$$

$$(2.25) \quad \zeta'_\epsilon := \text{id} + \epsilon(12) \quad , \quad \epsilon \in \{1, -1\}$$

$$(2.26) \quad \xi_\nu \mapsto \xi''_\nu \in \mathbb{K}[\mathcal{S}_5] \quad , \quad \eta \mapsto \eta'' \in \mathbb{K}[\mathcal{S}_5] .$$

Formula (2.26) denotes the embedding of $\xi_\nu, \eta \in \mathbb{K}[\mathcal{S}_3]$ into $\mathbb{K}[\mathcal{S}_5]$ which is induced by the mapping $\mathcal{S}_3 \rightarrow \mathcal{S}_5$, $[i_1, i_2, i_3] \mapsto [1, 2, i_1 + 2, i_2 + 2, i_3 + 2]$.

For expressions $y_{t'}^*(U \otimes S)$ and $y_{t'}^*(U \otimes A)$ we consider the $\mathbb{K}[\mathcal{S}_5]$ -elements

$$(2.27) \quad \sigma_{\nu, \epsilon} := y_{t'}^* \cdot \zeta''_\epsilon \cdot \xi'_\nu \quad , \quad \rho_\epsilon := y_{t'}^* \cdot \zeta''_\epsilon \cdot \eta'$$

$$(2.28) \quad \zeta''_\epsilon := \text{id} + \epsilon(45) \quad , \quad \epsilon \in \{1, -1\}$$

$$(2.29) \quad \xi_\nu \mapsto \xi'_\nu \in \mathbb{K}[\mathcal{S}_5] \quad , \quad \eta \mapsto \eta' \in \mathbb{K}[\mathcal{S}_5] .$$

and the embedding $\mathcal{S}_3 \rightarrow \mathcal{S}_5$, $[i_1, i_2, i_3] \mapsto [i_1, i_2, i_3, 4, 5]$ in (2.29).

Using the Mathematica package PERMS [10] we verified in [14] that

$$\boxed{\rho_\epsilon \neq 0 \quad \text{and} \quad \sigma_{\nu, \epsilon} \neq 0 \Leftrightarrow \nu \neq \frac{1}{2}}$$

in both cases. If $\rho_\epsilon \neq 0, \sigma_{\nu, \epsilon} \neq 0$ then the minimal right ideals $y_{t'}^* \cdot \mathbb{K}[\mathcal{S}_5]$, $\rho_\epsilon \cdot \mathbb{K}[\mathcal{S}_5]$ and $\sigma_{\nu, \epsilon} \cdot \mathbb{K}[\mathcal{S}_5]$ coincide, i.e. the symmetry operators $\rho_\epsilon, \sigma_{\nu, \epsilon}$ can be used to define the symmetry class of algebraic covariant derivative curvature tensors. A tensor $T \in \mathcal{T}_5 V$ is an algebraic covariant derivative curvature tensor iff a tensor $T' \in \mathcal{T}_5 V$ exists such that $T = \rho_\epsilon T'$ or $T = \sigma_{\nu, \epsilon} T'$ (if $\nu \neq \frac{1}{2}$).

On the basis of this fact the statement of Theorem 1.10 can be proved (see [14]). If $\nu = \frac{1}{2}$, then ξ_ν generates the right ideal $\mathfrak{r}_0 = \xi_\nu \cdot \mathbb{K}[\mathcal{S}_3]$. \mathfrak{r}_0 was excluded in Theorem 1.10 because $\sigma_{1/2, \epsilon} = 0$.

3. PROCEDURES FOR THE CONSTRUCTION OF SHORT COORDINATE REPRESENTATIONS OF THE TENSORS (1.19)

In this section we begin to construct short coordinate representations of the tensors (1.19). We use a common symbol W_{ij} for the coordinates A_{ij} and S_{ij} of the tensors $A, S \in \mathcal{T}_2 V$. Then the relations

$$(3.1) \quad S_{ij} = S_{ji} \quad , \quad A_{ij} = -A_{ji} .$$

can be written as

$$(3.2) \quad W_{ij} = \epsilon W_{ji} \quad , \quad \epsilon = \begin{cases} 1 & \text{if } W \text{ symmetric} \\ -1 & \text{if } W \text{ skew-symmetric} . \end{cases}$$

If we apply the symmetry operator $\frac{1}{24}y_{t'}^*$ to tensors $S \otimes U$, $A \otimes U$, $U \otimes S$ or $U \otimes A$, then we obtain long polynomials

$$(3.3) \quad \mathfrak{P}_{i_1 \dots i_5} = \sum_{p \in \mathcal{S}_5} c_p U_{i_{p(1)} i_{p(2)} i_{p(3)}} W_{i_{p(4)} i_{p(5)}} \quad , \quad c_p \in \mathbb{K}$$

in the coordinates of A , S and U . A *first reduction* of the number of summands of (3.3) can be determined by means of the relations (3.1). The results are the expressions which we present in Appendix A. They are polynomials consisting of 40 summands in general. In particular they fulfil

$$(3.4) \quad \mathfrak{P}_{i_1 \dots i_5} = \frac{1}{24}(y_{t'}^*(U \otimes W))_{ijklr} = \epsilon \frac{1}{24}(y_{t'}^*(W \otimes U))_{ijklr} ,$$

i.e. a permutation of U and W causes at most a change of the sign (see the lemma in Appendix A). Equation (3.4) yields statement (1) of Theorem 1.11.

A *second reduction* of the number of summands can be carried out by means of all linear identities which the coordinates of the tensor U satisfy. The determination of the set of these identities is based on the following method from [9, Sec. III.4.1, I.1.2] (see also [12]).

Linear identities for the coordinates $T_{i_1 \dots i_r}$ of a tensor $T \in \mathcal{T}_r V$ have the form

$$(3.5) \quad 0 = \sum_{p \in \mathcal{S}_r} x_p T_{i_{p(1)} \dots i_{p(r)}} \quad , \quad x_p \in \mathbb{K} ,$$

where x_p are given numbers from \mathbb{K} .

Assume that the symmetry class $\mathcal{T}_{\mathfrak{r}}$ of $T \in \mathcal{T}_r V$ is defined by a right ideal $\mathfrak{r} \subseteq \mathbb{K}[\mathcal{S}_r]$. Then we know from Proposition 2.4 that every group ring element $T_b \in \mathbb{K}[\mathcal{S}_r]$, $b \in V^r$, belongs to the left ideal $\mathfrak{l} = \mathfrak{r}^*$. Moreover, \mathfrak{l} is the smallest linear subspace of $\mathbb{K}[\mathcal{S}_r]$ which contains all T_b of all $T \in \mathcal{T}_{\mathfrak{r}}$ if $\dim V \geq r$ (see Proposition 2.5).

Let us denote by \mathfrak{l}^\perp the orthogonal subspace of \mathfrak{l} , i.e. the space of all linear functionals $\langle x, \cdot \rangle$ on $\mathbb{K}[\mathcal{S}_r]$ that vanish on all elements of \mathfrak{l} . Obviously, every $x \in \mathfrak{l}^\perp$ yields a linear identity for the coordinates of T since we can write

$$(3.6) \quad 0 = \langle x, T_b \rangle = \sum_{p \in \mathcal{S}_r} x_p T_b(p) = \sum_{p \in \mathcal{S}_r} x_p T_{i_{p(1)} \dots i_{p(r)}} ,$$

where $x_p := \langle x, p \rangle$, $p \in \mathcal{S}_r$. (The last step is correct if the b occurring in (3.6) is an r -tuple of basis vectors of V .)

Every identity (3.6) can be used to eliminate coordinates of T in a polynomial in tensor coordinates. If \mathfrak{l} is spanned by all T_b of the tensors considered, then \mathfrak{l}^\perp contains all linear identities which are possible between coordinates of T (compare Prop. 2.5).

We see immediately

Proposition 3.1. *If a basis $\{h_1, \dots, h_d\}$ of \mathfrak{l} is known, then the coefficients $x_p = \langle x, p \rangle$ of the $x \in \mathfrak{l}^\perp$ can be obtained from the linear equation system*

$$(3.7) \quad \langle x, h_i \rangle = \sum_{p \in \mathcal{S}_r} h_i(p) x_p = 0 \quad (i = 1, \dots, d).$$

The determination of a basis $\{h_1, \dots, h_d\}$ of \mathfrak{l} can be carried out efficiently by means of inverse Fourier transforms since there is a fast construction of bases in $\bigotimes_{\lambda \vdash r} \mathbb{K}^{d_\lambda \times d_\lambda}$. We present here only the construction of bases of minimal left ideals of a matrix ring $\mathbb{K}^{d \times d}$. See [9, Sec.I.1.2] for more general cases.

Definition 3.2. Let $a \in \mathbb{K}^d$ be a d -tuple and $i \in \mathbb{N}$ be a natural number with $1 \leq i \leq d$. Then we denote by $C_{i,a} \in \mathbb{K}^{d \times d}$ that matrix in which the i -th row is equal to a and all other rows are filled with zeros.

Proposition 3.3. *Let $A \in \mathbb{K}^{d \times d}$, $A \neq 0$, be a generating element of a minimal left ideal $\mathfrak{l} = \mathbb{K}^{d \times d} \cdot A$ of $\mathbb{K}^{d \times d}$. If $a \neq 0$ is a non-vanishing row of A , then the matrix set*

$$(3.8) \quad \mathcal{B} := \{ C_{i,a} \mid i = 1, \dots, d \}$$

is a basis of \mathfrak{l} .

Proposition 3.3 is a special case of [9, Prop.I.1.33] (compare also [9, Prop.I.1.35] or [12, Prop.5.1]). Due to Proposition 3.3 we can construct bases for Proposition 3.1 by the following

Procedure 3.4. Let $e \in \mathbb{K}[\mathcal{S}_r]$ be a generating idempotent of a d -dimensional, minimal right ideal $\mathfrak{r} \subset \mathbb{K}[\mathcal{S}_r]$ which defines a symmetry class $\mathcal{T}_{\mathfrak{r}}$ of tensors from $\mathcal{T}_r V$. Then we can obtain a basis $\{h_1, \dots, h_d\}$ for Proposition 3.1 by the following steps:

- (1) Calculate the generating idempotent e^* of the minimal left ideal $\mathfrak{l} = \mathfrak{r}^*$.
- (2) Form the discrete Fourier transform of e^* which possesses only one non-vanishing matrix block E , i.e.

$$D(e^*) = \begin{pmatrix} 0 & & \\ & E & \\ & & 0 \end{pmatrix}.$$

- (3) Search for a row $a \neq 0$ of E and form the basis $\mathcal{B} = \{C_{i,a} \mid i = 1, \dots, d\}$ of the left ideal $\mathbb{K}^{d \times d} \cdot E$.

(4) Calculate

$$h_i = D^{-1} \begin{pmatrix} 0 \\ C_{i,a} \\ 0 \end{pmatrix}.$$

Remark 3.5. In the \mathcal{S}_3 we could also determine a basis of $\mathfrak{l} = \mathbb{K}[\mathcal{S}_3] \cdot e^*$ by searching for linearly independent vectors in the set

$$(3.9) \quad \{p \cdot e^* \mid p \in \mathcal{S}_r\}.$$

However, this way is not efficient in a large \mathcal{S}_r since (3.9) leads to an $(r! \times r!)$ -matrix of coefficients. In a large \mathcal{S}_r Procedure 3.4 is better than an investigation of (3.9).

If we know a basis $\mathcal{B} = \{h_1, \dots, h_d\}$ of $\mathfrak{l} = \mathbb{K}[\mathcal{S}_r] \cdot e^*$, then we can determine solutions of the equation system (3.7) and identities (3.6) by the following obvious procedure:

Procedure 3.6. (1) Choose a subset $\mathcal{P} \subseteq \mathcal{S}_r$ of $d = \dim \mathfrak{l}$ permutations, such that $\{n_p := [h_i(p)]_{i=1, \dots, d} \mid p \in \mathcal{P}\}$ is a set of d linearly independent column vectors. Then the equation system (3.7) can be written in the form

$$(3.10) \quad 0 = \sum_{p \in \mathcal{P}} h_i(p) x_p + \sum_{p \in \mathcal{S}_r \setminus \mathcal{P}} h_i(p) x_p \quad (i = 1, \dots, d).$$

(2) For every fixed $\bar{p} \in \mathcal{S}_r \setminus \mathcal{P}$ we can determine a unique solution $x_p^{(\bar{p})}$ of (3.10) which fulfils

$$(3.11) \quad x_p^{(\bar{p})} = \begin{cases} 1 & \text{if } p = \bar{p} \\ 0 & \text{if } p \in \mathcal{S}_r \setminus (\mathcal{P} \cup \{\bar{p}\}) \end{cases}.$$

The $x_p^{(\bar{p})}$ with $p \in \mathcal{P}$ can be calculated from the equation system

$$(3.12) \quad 0 = \sum_{p \in \mathcal{P}} h_i(p) x_p + h_i(\bar{p}) \quad (i = 1, \dots, d).$$

Obviously, the set of all such solutions $\{x_p^{(\bar{p})} \mid p \in \mathcal{S}_r\}$, $\bar{p} \in \mathcal{S}_r \setminus \mathcal{P}$ is a basis of the null space of (3.10).

(3) Every above solution $\{x_p^{(\bar{p})} \mid p \in \mathcal{S}_r\}$, $\bar{p} \in \mathcal{S}_r \setminus \mathcal{P}$ defines a linear identity

$$(3.13) \quad 0 = \sum_{p \in \mathcal{P}} x_p^{(\bar{p})} T_{i_{p(1)} \dots i_{p(r)}} + T_{i_{\bar{p}(1)} \dots i_{\bar{p}(r)}} \quad (\bar{p} \in \mathcal{S}_r \setminus \mathcal{P})$$

for the coordinates of the tensors T of the symmetry class \mathcal{T}_* .

The identities (3.13) have the remarkable property that they depend only on the choice of the set \mathcal{P} .

Theorem 3.7. Let $\mathcal{B} = \{h_1, \dots, h_d\}$ and $\tilde{\mathcal{B}} = \{\tilde{h}_1, \dots, \tilde{h}_d\}$ be bases of the above d -dimensional left ideal \mathfrak{l} , which defines a symmetry class \mathcal{T}_* . Furthermore, let $\mathcal{P} \subseteq \mathcal{S}_r$ be a subset of d permutations such that $\{n_p = [h_i(p)]_{i=1, \dots, d} \mid p \in \mathcal{P}\}$ is a set of d linearly independent column vectors. Then the following holds:

- (1) The set $\{\tilde{n}_p = [\tilde{h}_i(p)]_{i=1,\dots,d} \mid p \in \mathcal{P}\}$ belonging to $\tilde{\mathcal{B}}$ also consists of d linearly independent column vectors.
- (2) For \mathcal{B} , $\tilde{\mathcal{B}}$ and \mathcal{P} the Procedure 3.6 yields sets of identities (3.13) whose coefficients satisfy

$$(3.14) \quad \forall p \in \mathcal{P} : \quad x_p^{(\bar{p})} = \tilde{x}_p^{(\bar{p})}.$$

Proof. Ad (1): The vectors h_i , \tilde{h}_i of \mathcal{B} , $\tilde{\mathcal{B}}$ satisfy a relation

$$(3.15) \quad \tilde{h}_i = \sum_{k=1}^d K_{ik} h_k$$

where $K := [K_{ik}]_{i,k=1,\dots,d}$ is a regular $d \times d$ -matrix. From (3.15) we obtain $\tilde{n}_p = K \cdot n_p$ for all $p \in \mathcal{P}$. Consequently the \tilde{n}_p are linearly independent again.

Ad (2): For all $\bar{p} \in \mathcal{S}_r \setminus \mathcal{P}$ the $\tilde{x}_p^{(\bar{p})}$ with $p \in \mathcal{P}$ can be calculated from the equation system

$$(3.16) \quad 0 = \sum_{p \in \mathcal{P}} \tilde{h}_i(p) \tilde{x}_p + \tilde{h}_i(\bar{p}) \quad (i = 1, \dots, d)$$

which is equivalent to

$$(3.17) \quad 0 = \sum_{k=1}^d K_{ik} \left\{ \sum_{p \in \mathcal{P}} h_k(p) \tilde{x}_p + h_k(\bar{p}) \right\} \quad (i = 1, \dots, d).$$

Since the \tilde{n}_p are linearly independent column vectors, system (3.16) has a unique solution. On the other hand, we see from (3.17) that the $x_p^{(\bar{p})}$ with $p \in \mathcal{P}$ solve (3.17) and (3.16). Thus (3.14) is correct. \square

Now we consider a finite set of $T^{(1)}, T^{(2)}, \dots, T^{(m)}$ of covariant tensors of orders r_1, r_2, \dots, r_m and a polynomial

$$(3.18) \quad \begin{aligned} \mathfrak{P}(T^{(1)}, \dots, T^{(m)})_{i_1 \dots i_r} &= \sum_{q \in \mathcal{S}_r} c_q T_{i_{q(1)} \dots i_{q(r_1)}}^{(1)} \cdot \dots \cdot T_{i_{q(r-r_m+1)} \dots i_{q(r)}}^{(m)} \\ &= \sum_{q \in \mathcal{S}_r} c_q (T^{(1)} \otimes \dots \otimes T^{(m)})_{i_{q(1)} \dots i_{q(r)}} \end{aligned}$$

where $r = r_1 + \dots + r_m$. We do not require that $c_q \neq 0$ for all $q \in \mathcal{S}_r$. Thus the summation can run through a subset of \mathcal{S}_r only. Furthermore we assume that a *lexicographic ordering* is defined for the index names in (3.18).

If we know identities of type (3.13) for a tensor $T^{(k)}$ in (3.18), then we can transform all coordinates $\{T_{j_{s(1)} \dots j_{s(r_k)}}^{(k)} \mid s \in \mathcal{S}_{r_k}\}$ which belong to the same set of index names $\{j_1, \dots, j_{r_k}\} \subset \{i_1, \dots, i_r\}$ into the set $\{T_{j_{p(1)} \dots j_{p(r_k)}}^{(k)} \mid p \in \mathcal{P}\}$ of $d = |\mathcal{P}|$ coordinates. Here we assume that $\{j_1, \dots, j_{r_k}\}$ is an arrangement of indices whose names form an increasing sequence according to the lexicographic ordering.

The identities (3.13) do not depend on the choice of a basis \mathcal{B} , but only on the choice of the set \mathcal{P} . We can carry out the above reduction

$$(3.19) \quad \{T_{j_{s(1)} \dots j_{s(r_k)}}^{(k)} \mid s \in \mathcal{S}_{r_k}\} \rightarrow \{T_{j_{p(1)} \dots j_{p(r_k)}}^{(k)} \mid p \in \mathcal{P}\}$$

of tensor coordinates for every set \mathcal{P} which satisfies the assumptions of Procedure 3.6. However, if we apply this reduction process to a polynomial (3.18) and reduce the set of coordinates of a single tensor $T^{(k)}$ in (3.18), then it may happen that we obtain different lengths of the transformed $\mathfrak{P}_{i_1 \dots i_r}$ for different sets \mathcal{P} . Consequently there is the problem to find such a set \mathcal{P} for which the transformed $\mathfrak{P}_{i_1 \dots i_r}$ has a minimal length.

In the treatment of polynomials (3.3) we have to carry out transformations (3.13) only with respect to the tensor U . The above discussion leads to the following

Procedure 3.8. Let $\mathfrak{l} = \mathbb{K}[\mathcal{S}_3] \cdot e^*$ be a minimal left ideal describing the symmetry class of the tensor U and $\mathcal{B} = \{h_1, h_2\}$ be a basis of \mathfrak{l} determined by Procedure 3.4. (The hook length formula tells us that $\dim \mathfrak{l} = 2$ for every minimal left ideal \mathfrak{l} belonging to the equivalence class of $(2\ 1) \vdash 3$.) Then carry out the following steps for every subset $\mathcal{P} \subset \mathcal{S}_3$ with $|\mathcal{P}| = 2$:

- (1) Check the condition

$$(3.20) \quad \Delta_{\mathcal{P}} := \det \left(h_i(p) \right)_{\substack{i=1,2 \\ p \in \mathcal{P}}} \neq 0.$$

If (3.20) is not valid then skip the steps (2) and (3) for the set \mathcal{P} .

- (2) If $\Delta_{\mathcal{P}} \neq 0$ then determine identities (3.13) for the coordinates of U by means of Procedure 3.6 from \mathcal{B} and \mathcal{P} .
- (3) Carry out a reduction (3.19) of the coordinates of U in (3.3) by means of the identities (3.13) from step (2). Determine the number of summands of the reduced polynomial (3.3).

Remark 3.9. It is clear that Procedure 3.8 can be generalized.

- (1) If a polynomial (3.18) with m tensors $T^{(k)}$ is given, then we could reduce the set of coordinates of every tensor $T^{(k)}$ in (3.18) by a procedure of the type 3.8.
- (2) An exact transfer of Procedure 3.8 to the general case (3.18) means that we determine identities (3.13) for a given set \mathcal{P} and use these fixed identities for all index sets $\{j_1, \dots, j_{r_k}\}$ of the considered tensor $T^{(k)}$ in the reduction process (3.19). We could also go a more general way. For every set of index names $\{j_1, \dots, j_{r_k}\}$ occuring in coordinates of the considered tensor $T^{(k)}$ we could run through all possible sets \mathcal{P} and search for such a \mathcal{P} for which the length of (3.18) becomes minimal. This modification of Procedure 3.8 could yield shorter polynomials (3.18) than the original Procedure 3.8.

In the present paper we use the Procedure 3.8 in the above form since the generalization Remark 3.9 (2) leads to a considerable increase of the expenditure of calculation. The results of the above Procedure 3.8 are polynomials $\mathfrak{P}(U, W)_{i_1 \dots i_5}$

of type (3.3) in which *the arrangement of the indices of U is defined by one and the same set \mathcal{P} for all sets $\{j_1, j_2, j_3\}$ of index names belonging to this tensor.*

4. DETERMINATION OF SHORT COORDINATE REPRESENTATIONS OF THE TENSORS (1.19)

Now we apply the Procedures 3.4, 3.8 and the Subprocedure 3.6 to the idempotents (2.21), (2.22) to obtain linear identities (3.13) for the tensors U and to determine the shortest polynomials (3.3) which can be constructed by these procedures. We carry out our calculations by means of PERMS [10]. The **Mathematica** notebooks of the computation can be downloaded from [7].

In all following calculations we make the *assumption* that

$$(4.1) \quad \boxed{\dim V \geq 3.}$$

Then the left ideals $\mathfrak{l} = \mathfrak{r}^*$ which belong to the symmetry classes of the tensors U are generated by all U_b of the U and the manifold of solutions of the corresponding equation system (3.7) defines a complete set of linear identities for the tensors U (see Proposition 2.5).

In our calculations we use the following numbering of the permutations of \mathcal{S}_3 .

$$(4.2) \quad \begin{array}{c|c|c|c|c|c|c} i & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline p_i & [1, 2, 3] & [1, 3, 2] & [2, 1, 3] & [2, 3, 1] & [3, 1, 2] & [3, 2, 1] \end{array}.$$

4.1. Short formulas with a tensor U defined by η . The hook lenght formula tells us that the idempotent η given in (2.22) generates a 2-dimensional right ideal. If we apply Procedure 3.4 to η , we obtain

$$\begin{aligned} \eta^* &= \frac{1}{3} \{[1, 2, 3] - [2, 1, 3] - [3, 1, 2] + [3, 2, 1]\} \\ D(\eta^*) &= \begin{pmatrix} 0 & & \\ & E & \\ & & 0 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 0 & & \\ & -1 & 2 \\ & -2 & 4 \\ & & & 0 \end{pmatrix}. \end{aligned}$$

The first row $a = \frac{1}{3}(-1, 2)$ of E leads to the basis

$$C_{1,a} = \frac{1}{3} \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix}, \quad C_{2,a} = \frac{1}{3} \begin{pmatrix} 0 & 0 \\ -1 & 2 \end{pmatrix}$$

of $\mathbb{K}^{2 \times 2} \cdot E$ and an inverse Fourier transform yields the basis

$$\begin{aligned} (4.3) \quad h_1 &= \frac{1}{9} \{-[1, 2, 3] + 2[1, 3, 2] - [2, 1, 3] + 2[2, 3, 1] - [3, 1, 2] - [3, 2, 1]\} \\ h_2 &= \frac{1}{9} \{2[1, 2, 3] - [1, 3, 2] - [2, 1, 3] - [2, 3, 1] - [3, 1, 2] + 2[3, 2, 1]\} \end{aligned}$$

of $\mathfrak{l} = \mathbb{K}[\mathcal{S}_3] \cdot \eta^*$.

The coefficients of the group ring elements (4.3) form the rows of the coefficient matrix of the system (3.7). From (4.3) we obtain the matrix

$$(4.4) \quad \frac{1}{9} \begin{pmatrix} -1 & 2 & -1 & 2 & -1 & -1 \\ 2 & -1 & -1 & -1 & -1 & 2 \end{pmatrix}.$$

Let us carry out a step of Procedure 3.8 for the matrix (4.4). We start with the set $\mathcal{P} = \{[1, 2, 3], [1, 3, 2]\}$ which is connected with the first two columns of (4.4). Obviously we have $\Delta_{\mathcal{P}} \neq 0$. The Gaussian algorithm transforms (4.4) into

$$(4.5) \quad \begin{pmatrix} 1 & 0 & -1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 1 & -1 & 0 \end{pmatrix}.$$

The null space of (4.5) has the basis

$$(4.6) \quad \begin{pmatrix} -1, 0, 0, 0, 0, 1 \\ 0, -1, 0, 1, 0, 0 \end{pmatrix} \begin{pmatrix} 1, 1, 0, 0, 1, 0 \\ 1, 1, 1, 0, 0, 0 \end{pmatrix}.$$

If we use the vectors (4.6) to define linear identities (3.13) for the coordinates of $U \in \mathcal{T}_{\Gamma^*}$ we obtain

$$(4.7) \quad \begin{array}{rclcl} - & U_{ijk} & + & & U_{kji} & = & 0 \\ & U_{ijk} & + & U_{ikj} & + & U_{kij} & = & 0 \\ & & - & U_{ikj} & + & U_{jki} & = & 0 \\ & U_{ijk} & + & U_{ikj} & + & U_{jik} & = & 0 \end{array}.$$

Using the identities (4.7) we can express all coordinates of the tensor U by the coordinates U_{ijk} and U_{ikj} which possess the two lexicographically smallest index arrangements (defined by \mathcal{P}). If we carry out the reduction process (3.19) for

$$(4.8) \quad \mathfrak{P}_{i_1 \dots i_5}^{\text{red}} := \frac{1}{24} (y_{\nu}^*(U \otimes W))_{ijklr} = \epsilon \frac{1}{24} (y_{\nu}^*(W \otimes U))_{ijklr}$$

by means of (4.7), then we obtain reduced polynomials $\mathfrak{P}_{i_1 \dots i_5}^{\text{red}}$ of length 12 if $\epsilon = 1$ and of length 20 if $\epsilon = -1$.

Table 1 shows the results of the calculations for the remaining 14 sets \mathcal{P} . In the first column a set \mathcal{P} is denoted by two numbers mn if \mathcal{P} contains the m -th and the n -th permutation from (4.2). We see that different lengths of $\mathfrak{P}_{i_1 \dots i_5}^{\text{red}}$ occur for symmetric tensors W whereas an alternating tensor W leads only to a length 20. In Appendix B we present the formulas for $\mathfrak{P}_{i_1 \dots i_5}^{\text{red}}$ under the conditions of the first row of Table 1 (i.e. $\mathcal{P} = \{[1, 2, 3], [1, 3, 2]\}$ and $\epsilon = \pm 1$). The computer calculations for Table 1 and Appendix B can be found in the **Mathematica** notebooks `part12b.nb`, `...`, `part56b.nb` in [7].

4.2. Short formulas with a tensor U defined by ξ_{ν} . Now we search for short formulas $\mathfrak{P}_{i_1 \dots i_5}^{\text{red}}$ which can be formed by means of a tensor U from the symmetry class defined by the idempotent ξ_{ν} (see (2.21)).

Again, the hook length formula tells us that the idempotent ξ_{ν} generates a 2-dimensional right ideal of $\mathbb{K}[\mathcal{S}_3]$ for all $\nu \in \mathbb{K}$. If we start Procedure 3.4 for ξ_{ν} , we

\mathcal{P}	$\Delta_{\mathcal{P}}$	length of $\mathfrak{P}_{i_1 \dots i_5}^{\text{red}}$	
		$\epsilon = 1$	$\epsilon = -1$
12	$\neq 0$	12	20
13	$\neq 0$	16	20
14	$\neq 0$	12	20
15	$\neq 0$	16	20
16	0		
23	$\neq 0$	12	20
24	0		
25	$\neq 0$	12	20
26	$\neq 0$	12	20
34	$\neq 0$	12	20
35	0		
36	$\neq 0$	16	20
45	$\neq 0$	12	20
46	$\neq 0$	12	20
56	$\neq 0$	16	20

TABLE 1. The lengths of $\mathfrak{P}_{i_1 \dots i_5}^{\text{red}}$ for an U from the symmetry class given by η .

obtain

$$\begin{aligned} \xi_{\nu}^* &= \frac{1}{3} \{ [1, 2, 3] + \nu [1, 3, 2] + (1 - \nu) [2, 1, 3] \\ &\quad + (-1 + \nu) [2, 3, 1] - \nu [3, 1, 2] - [3, 2, 1] \} \\ D(\xi_{\nu}^*) &= \begin{pmatrix} 0 & & \\ & E & \\ & & 0 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 0 & & & \\ & 4 - 2\nu & -2 + 4\nu & \\ & 2 - \nu & -1 + 2\nu & \\ & & & 0 \end{pmatrix}. \end{aligned}$$

It is easy to see that the first row of E is unequal to zero for all $\nu \in \mathbb{K}$. The condition $4 - 2\nu = 0$ leads to $\nu = 2$. However, we obtain $-2 + 4\nu = 6$ for $\nu = 2$. Thus the first row a of E can be used for the construction of a basis \mathcal{B} of the left ideal $\mathbb{K}^{2 \times 2} \cdot E$ according step (3) of Procedure 3.4. We obtain

$$C_{1,a} = \frac{1}{3} \begin{pmatrix} 4 - 2\nu & -2 + 4\nu \\ 0 & 0 \end{pmatrix}, \quad C_{2,a} = \frac{1}{3} \begin{pmatrix} 0 & 0 \\ 4 - 2\nu & -2 + 4\nu \end{pmatrix}.$$

Now an inverse Fourier transform yields the basis

$$\begin{aligned} h_1 &= \frac{1}{9} \{ (4 - 2\nu) [1, 2, 3] + (-2 + 4\nu) [1, 3, 2] + (4 - 2\nu) [2, 1, 3] \\ &\quad + (-2 + 4\nu) [2, 3, 1] - (2 + 2\nu) [3, 1, 2] - (2 + 2\nu) [3, 2, 1] \} \\ h_2 &= \frac{1}{9} \{ (-2 + 4\nu) [1, 2, 3] + (4 - 2\nu) [1, 3, 2] - (2 + 2\nu) [2, 1, 3] \\ &\quad - (2 + 2\nu) [2, 3, 1] + (4 - 2\nu) [3, 1, 2] + (-2 + 4\nu) [3, 2, 1] \} \end{aligned} \quad (4.9)$$

of $\mathfrak{l} = \mathbb{K}[\mathcal{S}_3] \cdot \eta^*$.

Again, the coefficients of the group ring elements (4.9) form the rows of the coefficient matrix of the system (3.7). From (4.9) we obtain the matrix

$$(4.10) \quad \frac{1}{9} \begin{pmatrix} 4-2\nu & -2+4\nu & 4-2\nu & -2+4\nu & -2-2\nu & -2-2\nu \\ -2+4\nu & 4-2\nu & -2-2\nu & -2-2\nu & 4-2\nu & -2+4\nu \end{pmatrix}.$$

Since all elements of the matrix (4.10) depend linearly on ν , every determinant $\Delta_{\mathcal{P}}$ is a quadratic polynomial in ν . Our computer calculations show that $\Delta_{\mathcal{P}} \neq 0$ for all 15 sets $\mathcal{P} \subset \mathcal{S}_3$ with $|\mathcal{P}| = 2$. Consequently, there are at most two roots $\nu_1, \nu_2 \in \mathbb{K}$ for a fixed \mathcal{P} such that $\Delta_{\mathcal{P}}(\nu_i) = 0$.

4.2.1. The generic case. Let us first investigate the *generic case*. For every \mathcal{P} we restrict us to such values $\nu \in \mathbb{K}$ for which $\Delta_{\mathcal{P}}(\nu) \neq 0$. Then we can carry out the Procedures 3.6 and 3.8. Obviously the coefficients of the identities (3.13) and of the reduced polynomials $\mathfrak{P}_{i_1 \dots i_5}^{\text{red}}$ will then be rational functions in ν . More precisely, for every set $\mathcal{P} \subset \mathcal{S}_3$ with $|\mathcal{P}| = 2$ there are polynomials

$$P_q^{\mathcal{P}}(\nu), Q_q^{\mathcal{P}}(\nu) \in \mathbb{K}[\nu] \quad , \quad q \in \mathcal{S}_5,$$

such that the result of Procedure 3.8 belonging to \mathcal{P} can be written as

$$(4.11) \quad \mathfrak{P}_{i_1 \dots i_5}^{\text{red}} = \sum_{q \in \mathcal{S}_5} \frac{P_q^{\mathcal{P}}(\nu)}{Q_q^{\mathcal{P}}(\nu)} U_{i_{q(1)} i_{q(2)} i_{q(3)}} W_{i_{q(4)} i_{q(5)}}.$$

Remark 4.1. Note that the calculation of (4.11) is possible for all values $\nu \in \mathbb{K}$ which fulfill $\Delta_{\mathcal{P}}(\nu) \neq 0$. In particular the polynomials $Q_q^{\mathcal{P}}(\nu)$ in (4.11) satisfy $Q_q^{\mathcal{P}}(\nu) \neq 0$ for all ν with $\Delta_{\mathcal{P}}(\nu) \neq 0$. If we would use *Cramer's rule* to solve the equation system (3.10) and to determine the linear identities (3.13), we could write $Q_q^{\mathcal{P}}(\nu) = \Delta_{\mathcal{P}}(\nu)$ for all $Q_q^{\mathcal{P}}(\nu)$. However, we give up a condition such as $Q_q^{\mathcal{P}}(\nu) = \Delta_{\mathcal{P}}(\nu)$ to allow a reduction of the fractions $P_q^{\mathcal{P}}(\nu)/Q_q^{\mathcal{P}}(\nu)$.

Let us carry out Procedure 3.8 for the set $\mathcal{P} = \{[1, 2, 3], [1, 3, 2]\}$ which characterizes the first two columns of (4.10). In this case we get the determinant

$$(4.12) \quad \Delta_{\mathcal{P}}(\nu) = \frac{4}{27} (1 - \nu)(1 + \nu).$$

(4.12) has the roots $\nu_1 = 1$ and $\nu_2 = -1$. For $\nu \notin \{1, -1\}$ Procedure 3.6 yields now the identities

$$(4.13) \quad \begin{array}{llll} -\frac{\nu^2-\nu+1}{\nu^2-1} U_{ijk} & + & \frac{2\nu-1}{\nu^2-1} U_{ikj} & + & U_{kji} & = & 0 \\ \frac{\nu^2-2\nu}{\nu^2-1} U_{ijk} & + & \frac{\nu^2-\nu+1}{\nu^2-1} U_{ikj} & + & U_{kij} & = & 0 \\ \frac{2\nu-1}{\nu^2-1} U_{ijk} & - & \frac{\nu^2-\nu+1}{\nu^2-1} U_{ikj} & + & U_{jki} & = & 0 \\ \frac{\nu^2-\nu+1}{\nu^2-1} U_{ijk} & + & \frac{\nu^2-2\nu}{\nu^2-1} U_{ikj} & + & U_{jik} & = & 0 \end{array}.$$

If we apply the identities (4.13) to reduce the polynomials $\mathfrak{P}_{i_1 \dots i_5}$ in Appendix A, we obtain reduced polynomials $\mathfrak{P}_{i_1 \dots i_5}^{\text{red}}$ of length 16 if W is symmetric and of length 20 if W is skew-symmetric. (These numbers are only correct if we consider the generic case in which all polynomials $P_q^{\mathcal{P}}(\nu)$ in (4.11) do not vanish.) Explicite formulas for the reduced polynomials $\mathfrak{P}_{i_1 \dots i_5}^{\text{red}}$ are presented in Appendix C. The computer

\mathcal{P}	roots of $\Delta_{\mathcal{P}}(\nu)$	length of $\mathfrak{P}_{i_1 \dots i_5}^{\text{red}}$	
		$\epsilon = 1$	$\epsilon = -1$
12	1, -1	16	20
13	0, 2	16	20
14	$e^{i\pi/3}, e^{-i\pi/3}$	16	20
15	$e^{i\pi/3}, e^{-i\pi/3}$	16	20
16	1/2	16	20
23	$e^{i\pi/3}, e^{-i\pi/3}$	16	20
24	1/2	16	20
25	0, 2	16	20
26	$e^{i\pi/3}, e^{-i\pi/3}$	16	20
34	1, -1	16	20
35	1/2	16	20
36	$e^{i\pi/3}, e^{-i\pi/3}$	16	20
45	$e^{i\pi/3}, e^{-i\pi/3}$	16	20
46	0, 2	16	20
56	1, -1	16	20

TABLE 2. The lengths of $\mathfrak{P}_{i_1 \dots i_5}^{\text{red}}$ for an U from the symmetry class given by ξ_ν in the generic case $\Delta_{\mathcal{P}}(\nu) \neq 0$ and $P_q^{\mathcal{P}}(\nu) \neq 0$ for all $q \in \mathcal{S}_5$.

calculations can be found in the **Mathematica** notebooks [7, proc44.nb, part12a.nb].

When we repeat this computation for the remaining 14 sets \mathcal{P} , we obtain the results given in Table 2. The second column of Table 2 contains the roots of $\Delta_{\mathcal{P}}(\nu)$ which we have to exclude. The computer calculations for Table 2 can be found in the **Mathematica** notebooks `part12a.nb`, ..., `part56a.nb` in [7].

Remark 4.2. Note that some $\Delta_{\mathcal{P}}(\nu)$ possess only the root $\nu = \frac{1}{2}$ which is the critical ν -value for the construction of algebraic covariant derivative curvature tensors. Since all $\nu \neq \frac{1}{2}$ are allowed values in (4.11) for such sets \mathcal{P} , the $\mathfrak{P}_{i_1 \dots i_5}^{\text{red}}$ of such \mathcal{P} are formulas which represent algebraic covariant derivative curvature tensors for every $\nu \neq \frac{1}{2}$. As an example, we present in Appendix C formulas of $\mathfrak{P}_{i_1 \dots i_5}^{\text{red}}$ for the set \mathcal{P} denoted by "16" in Table 2.

Remark 4.3. We see that the lengths of $\mathfrak{P}_{i_1 \dots i_5}^{\text{red}}$ are independent on \mathcal{P} in the generic case. This property is not self-evident. The following considerations show that our calculation could just as well result in contrary findings.

We denote by $I := \{i, j, k, l, r\}$ the set of index names used for the polynomials of tensor coordinates in the Appendices and by ' \prec ' the lexicographic order for these index names. Let \mathfrak{I} be the set of all pairs $(j_1, j_2) \in I \times I$ with $j_1 \prec j_2$. For every $(j_1, j_2) \in \mathfrak{I}$ we denote by i_1, i_2, i_3 the elements of $I \setminus \{j_1, j_2\}$ where we assume $i_1 \prec i_2 \prec i_3$.

Using these notations we can write the expression for $\frac{1}{24}(y_{t'}^*(U \otimes W))_{ijklr}$ given in Appendix A in the form

$$(4.14) \quad \mathfrak{P}_{ijklr} := \frac{1}{24}(y_{t'}^*(U \otimes W))_{ijklr} = \sum_{(j_1, j_2) \in \mathfrak{J}} \sum_{s \in \mathcal{S}_3} A_{\epsilon, s}^{(j_1, j_2)} U_{i_{s(1)} i_{s(2)} i_{s(3)}} W_{j_1 j_2},$$

where $A_{\epsilon, s}^{(j_1, j_2)} \in \mathbb{K}$ are constant coefficients. (If the summation with respect to s runs only through a subset of \mathcal{S}_3 , then some of these coefficients vanish).

If a set $\mathcal{P} \subset \mathcal{S}_3$ of two permutations is given, then we can express the linear identities (3.13) for U by

$$(4.15) \quad U_{i_{s(1)} i_{s(2)} i_{s(3)}} = \sum_{p \in \mathcal{P}} B_{sp}^{\mathcal{P}}(\nu) U_{i_{p(1)} i_{p(2)} i_{p(3)}} \quad , \quad s \in \mathcal{S}_3 \setminus \mathcal{P}.$$

Here $B_{sp}^{\mathcal{P}}(\nu)$ are rational functions of ν .

When we set (4.15) into (4.14), we obtain

$$\begin{aligned} \mathfrak{P}_{ijklr}^{\text{red}} &= \sum_{(j_1, j_2) \in \mathfrak{J}} \sum_{p \in \mathcal{P}} A_{\epsilon, p}^{(j_1, j_2)} U_{i_{p(1)} i_{p(2)} i_{p(3)}} W_{j_1 j_2} + \\ &\quad \sum_{(j_1, j_2) \in \mathfrak{J}} \sum_{s \in \mathcal{S}_3 \setminus \mathcal{P}} \sum_{p' \in \mathcal{P}} A_{\epsilon, s}^{(j_1, j_2)} B_{sp'}^{\mathcal{P}}(\nu) U_{i_{p'(1)} i_{p'(2)} i_{p'(3)}} W_{j_1 j_2} \\ &= \sum_{(j_1, j_2) \in \mathfrak{J}} \sum_{p \in \mathcal{P}} \left\{ A_{\epsilon, p}^{(j_1, j_2)} + \sum_{s \in \mathcal{S}_3 \setminus \mathcal{P}} A_{\epsilon, s}^{(j_1, j_2)} B_{sp}^{\mathcal{P}}(\nu) \right\} U_{i_{p(1)} i_{p(2)} i_{p(3)}} W_{j_1 j_2} \\ &= \sum_{(j_1, j_2) \in \mathfrak{J}} \sum_{p \in \mathcal{P}} C_{\epsilon, p}^{(j_1, j_2), \mathcal{P}}(\nu) U_{i_{p(1)} i_{p(2)} i_{p(3)}} W_{j_1 j_2} \end{aligned}$$

where we use the notation

$$(4.16) \quad C_{\epsilon, p}^{(j_1, j_2), \mathcal{P}}(\nu) := A_{\epsilon, p}^{(j_1, j_2)} + \sum_{s \in \mathcal{S}_3 \setminus \mathcal{P}} A_{\epsilon, s}^{(j_1, j_2)} B_{sp}^{\mathcal{P}}(\nu).$$

If all rational functions $C_{\epsilon, p}^{(j_1, j_2), \mathcal{P}}(\nu)$ fulfill $C_{\epsilon, p}^{(j_1, j_2), \mathcal{P}}(\nu) \neq 0$, then we can consider the generic case of all such ν for which all $C_{\epsilon, p}^{(j_1, j_2), \mathcal{P}}(\nu)$ does not vanish. In this case we have

$$\text{length } \mathfrak{P}_{ijklr}^{\text{red}} = |\mathfrak{J}| \times |\mathcal{P}| = 10 \times 2 = 20.$$

Obviously, such a situation exists if $\epsilon = -1$.

However, a short look at (4.16) shows that some of the $C_{\epsilon, p}^{(j_1, j_2), \mathcal{P}}(\nu)$ could satisfy a relation $C_{\epsilon, p}^{(j_1, j_2), \mathcal{P}}(\nu) \equiv 0$ if the $A_{\epsilon, p}^{(j_1, j_2)}$ and $B_{sp}^{\mathcal{P}}(\nu)$ fulfill certain conditions. A trivial example yields the case $\epsilon = 1$. In this case we read from the expression $\frac{1}{24}(y_{t'}^*(U \otimes W))_{ijklr}$ in Appendix A that $A_{\epsilon, p}^{(j_1, j_2)} = 0$ if $(j_1, j_2) = (i, j)$ or $(j_1, j_2) = (k, l)$. This leads to $C_{\epsilon, p}^{(j_1, j_2), \mathcal{P}}(\nu) \equiv 0$ for $(j_1, j_2) = (i, j)$ or $(j_1, j_2) = (k, l)$ and to the reduction of $\text{length } \mathfrak{P}_{ijklr}^{\text{red}}$ from 20 to 16 summands.

Assume now that there would be two sets $\mathcal{P}_1, \mathcal{P}_2 \subset \mathcal{S}_3$ of two permutations to which different numbers of $C_{\epsilon, p}^{(j_1, j_2), \mathcal{P}_i}(\nu)$ with $C_{\epsilon, p}^{(j_1, j_2), \mathcal{P}_i}(\nu) \neq 0$ belong. Then we

would observe a change of length $\mathfrak{P}_{ijklr}^{\text{red}}$ even in the generic case if we pass from \mathcal{P}_1 to \mathcal{P}_2 .

4.2.2. The non-generic cases. Now we handel non-generic cases of the search for short formulas for length $\mathfrak{P}_{ijklr}^{\text{red}}$. We start with the expressions (4.11) which we found in the investigation of the generic case in Section 4.2.1 and carry out the following

Procedure 4.4. Consider the formula (4.11) which was won for a fixed set $\mathcal{P} = \{p_1, p_2\} \subset \mathcal{S}_3$ in the generic case.

- (1) Determine the set $N_{\mathcal{P}}$ of all $\nu \in \mathbb{K}$ for which at least one of the polynomials $P_q^{\mathcal{P}}(\nu)$ from (4.11) vanishes.
- (2) Remove all such $\nu \in N_{\mathcal{P}}$ from $N_{\mathcal{P}}$ that are roots of $\Delta_{\mathcal{P}}(\nu)$. Further remove $\nu = \frac{1}{2}$ from $N_{\mathcal{P}}$. (Compare Remark 4.1.)
- (3) Set every $\nu \in N_{\mathcal{P}}$ into the formula (4.11) belonging to \mathcal{P} and determine the number of summands in the resulting expression.

The results of PERMS-calculations according to Procedure 4.4 are listed in Tables 3 and 4. The **Mathematica** notebooks of these calculations are `roots12a.nb`, `...`, `roots56a.nb` in [7].

A summary is given in Table (4.17). This Table shows the minimal lengths of $\mathfrak{P}_{i_1 \dots i_5}^{\text{red}}$ and the elements of the set $N := \bigcup_{|\mathcal{P}|=2} N_{\mathcal{P}}$ which is the collection of all (allowed) roots of the polynomials $P_q^{\mathcal{P}}(\nu)$.

(4.17)

	$\epsilon = 1$	$\epsilon = -1$
min length $\mathfrak{P}_{i_1 \dots i_5}^{\text{red}}$	12	10
N	$\{-1, 0, 1, 2\}$	$\{-1, 2\}$

For every $\nu \in N$ there is a set \mathcal{P} such that $\mathfrak{P}_{i_1 \dots i_5}^{\text{red}}$ belonging to \mathcal{P} has the minimal length given in (4.17). Note, however, that the roots $\nu = 0$ and $\nu = 1$ occur only for symmetric tensors W (i.e. $\epsilon = 1$).

It is remarkable that for $\nu = -1$ and $\nu = 2$ sets \mathcal{P} can be found such that $\mathfrak{P}_{i_1 \dots i_5}^{\text{red}}$ has minimal length both for $\epsilon = 1$ and for $\epsilon = -1$. An example is the set $\mathcal{P} = \{[1, 2, 3], [3, 2, 1]\}$ denoted by "16". The expressions $\mathfrak{P}_{i_1 \dots i_5}^{\text{red}}$ of minimal length belonging to "16" are presented in Appendix D. The example of the set "16" yields the proof of statement 3 in Theorem 1.11.

5. INDEX COMMUTATION SYMMETRIES

In this section we investigate the question whether the tensors $U \in \mathcal{T}_3 V$ belonging to η or ξ_{ν} possess index commutation symmetries. We will see that such symmetries occur only for η and a finite set of ν -values. In particular the $\nu \in N$ given in (4.17) are such ν -values. This shows an interesting connection between index commutation symmetries and the maximal reduction of the length of $\mathfrak{P}_{i_1 \dots i_5}^{\text{red}}$.

\mathcal{P}	$\epsilon = 1$		$\epsilon = -1$	
	roots of $P_q^{\mathcal{P}}(\nu)$ with $\Delta_{\mathcal{P}}(\nu) \neq 0, \nu \neq 1/2$	length of $\mathfrak{P}_{i_1 \dots i_5}^{\text{red}}$	roots of $P_q^{\mathcal{P}}(\nu)$ with $\Delta_{\mathcal{P}}(\nu) \neq 0, \nu \neq 1/2$	length of $\mathfrak{P}_{i_1 \dots i_5}^{\text{red}}$
12	0	12		
	2	14	2	12
13	-1	14	-1	18
14	-1	14	-1	18
	0	12		
	2	12	2	10
15	-1	12	-1	10
	0	12		
	1	12		
	2	14	2	12
16	-1	12	-1	10
	0	12		
	1	12		
	2	12	2	10
23	-1	14	-1	18
	0	12		
	2	14	2	12
24	-1	14	-1	18
	2	14	2	18
25	-1	12	-1	10
	1	12		

TABLE 3. The lengths of $\mathfrak{P}_{i_1 \dots i_5}^{\text{red}}$ for an U from a ξ_ν -symmetry class, where ν is an allowed root of a $P_q^{\mathcal{P}}(\nu)$.

Definition 5.1. Let $C \subseteq \mathcal{S}_r$ be a subgroup of \mathcal{S}_r and $\theta : C \rightarrow \mathbb{K}^\times$ be a homomorphism of C onto a finite subgroup of the multiplicative subgroup $\mathbb{K}^\times := \mathbb{K} \setminus \{0\}$ of \mathbb{K} . We say that a tensor $T \in \mathcal{T}_r V$ possesses the *index commutation symmetry* (C, θ) iff

$$(5.1) \quad \forall c \in C : \quad cT = \theta(c)T.$$

We showed in [9, p.115]

Lemma 5.2. A tensor $T \in \mathcal{T}_r V$ possesses the symmetry (C, θ) iff T satisfies

$$(5.2) \quad \tilde{\theta}^* T = T$$

where

$$(5.3) \quad \tilde{\theta} := \frac{1}{|C|} \sum_{c \in C} \theta(c) c$$

is the normalized symmetrizer belonging to (C, θ) .

\mathcal{P}	$\epsilon = 1$		$\epsilon = -1$	
	roots of $P_q^{\mathcal{P}}(\nu)$ with $\Delta_{\mathcal{P}}(\nu) \neq 0, \nu \neq 1/2$	length of $\mathfrak{P}_{i_1 \dots i_5}^{\text{red}}$	roots of $P_q^{\mathcal{P}}(\nu)$ with $\Delta_{\mathcal{P}}(\nu) \neq 0, \nu \neq 1/2$	length of $\mathfrak{P}_{i_1 \dots i_5}^{\text{red}}$
26	-1	12	-1	10
	1	12		
	2	14	2	18
34	0	12		
	2	12	2	10
35	-1	14	-1	12
	0	12		
	1	12		
	2	14	2	12
36	-1	14	-1	12
	0	12		
	1	12		
	2	12	2	10
45	-1	14	-1	12
	1	12		
	2	14	2	18
46	-1	14	-1	12
	1	12		
56	2	14	2	18

TABLE 4. Continuation: The lengths of $\mathfrak{P}_{i_1 \dots i_5}^{\text{red}}$ for an U from a ξ_ν -symmetry class, where ν is an allowed root of a $P_q^{\mathcal{P}}(\nu)$.

Every symmetrizer (5.3) is an idempotent (see [9, p.115]). Lemma 5.2 means that a tensor $T \in \mathcal{T}_r V$ has a commutation symmetry (C, θ) iff T belongs to the symmetry class defined by the right ideal $\tilde{\mathfrak{r}} := \tilde{\theta}^* \cdot \mathbb{K}[\mathcal{S}_r]$.

Definition 5.3. Let $\mathfrak{r} \subseteq \mathbb{K}[\mathcal{S}_r]$ be a right ideal defining a symmetry class of tensors $T \in \mathcal{T}_r V$. We say that \mathfrak{r} admits a commutation symmetry (C, θ) over a subgroup $C \subseteq \mathcal{S}_r$ iff

$$(5.4) \quad \mathfrak{r} \subseteq \tilde{\mathfrak{r}} = \tilde{\theta}^* \cdot \mathbb{K}[\mathcal{S}_r].$$

The validity of a condition (5.4) can be checked comfortably by

Lemma 5.4.¹¹ Let $e, \tilde{e} \in \mathbb{K}[\mathcal{S}_r]$ be generating idempotents of right ideals $\mathfrak{r}, \tilde{\mathfrak{r}} \subseteq \mathbb{K}[\mathcal{S}_r]$. Then it holds

$$(5.5) \quad \tilde{\mathfrak{r}} \supseteq \mathfrak{r} \Leftrightarrow \tilde{e} \cdot e = e.$$

Consequently, we have to check the conditions

$$(5.6) \quad \tilde{\theta}^* \cdot \eta = \eta \quad \text{or} \quad \tilde{\theta}^* \cdot \xi_\nu = \xi_\nu$$

¹¹See H. Boerner [2, pp.54-59].

C	$ C $	$ C \cdot \tilde{\theta}$	name
$\langle [2, 1, 3] \rangle$	2	$[1, 2, 3] + [2, 1, 3]$	sym2a
$\langle [3, 2, 1] \rangle$	2	$[1, 2, 3] + [3, 2, 1]$	sym2b
$\langle [1, 3, 2] \rangle$	2	$[1, 2, 3] + [1, 3, 2]$	sym2c
$\langle [2, 1, 3] \rangle$	2	$[1, 2, 3] - [2, 1, 3]$	alt2a
$\langle [3, 2, 1] \rangle$	2	$[1, 2, 3] - [3, 2, 1]$	alt2b
$\langle [1, 3, 2] \rangle$	2	$[1, 2, 3] - [1, 3, 2]$	alt2c
\mathcal{A}_3	3	$[1, 2, 3] + e^{2i\pi/3}[2, 3, 1] + e^{-2i\pi/3}[3, 1, 2]$	z1
\mathcal{A}_3	3	$[1, 2, 3] + e^{-2i\pi/3}[2, 3, 1] + e^{2i\pi/3}[3, 1, 2]$	z2
\mathcal{A}_3	3	$[1, 2, 3] + [2, 3, 1] + [3, 1, 2]$	sym3
\mathcal{S}_3	6	$[1, 2, 3] + [1, 3, 2] + [2, 1, 3] + [2, 3, 1] + [3, 1, 2] + [3, 2, 1]$	sym6
\mathcal{S}_3	6	$[1, 2, 3] - [1, 3, 2] - [2, 1, 3] + [2, 3, 1] + [3, 1, 2] - [3, 2, 1]$	alt6

TABLE 5. All index commutation symmetries of subgroups of \mathcal{S}_3

if we want to find out whether a tensor $U \in \mathcal{T}_3 V$ from a symmetry class defined by η or ξ_ν admits a commutation symmetry (C, θ) , $C \subseteq \mathcal{S}_3$.

In Table 5 we give a complete list of all index commutation symmetries that can be defined on arbitrary subgroups $C \subseteq \mathcal{S}_3$. This list can be taken from the list [9, pp.179] of all index commutation symmetries belonging to subgroups of \mathcal{S}_r with $r \leq 6$. (See also the [HTML version](http://home.t-online.de/home/Bernd.Fiedler.RoschStr.Leipzig/tensym.htm)

home.t-online.de/home/Bernd.Fiedler.RoschStr.Leipzig/tensym.htm

of the Table from [9].)

Now a computer calculation by means of PERMS [10] leads to the following Theorem 5.5. The **Mathematica** notebook of the calculation can be found in [7, comsym.nb].

Theorem 5.5. *We consider the idempotents $\tilde{\theta}$ from Table 5 and the idempotents e from the set $\{\eta\} \cup \{\xi_\nu \mid \nu \in \mathbb{C}\}$. Then a condition $\tilde{\theta}^* \cdot e = e$ is satisfied iff the pair $(\tilde{\theta}, e)$ is one of the pairs from the following table*

$\tilde{\theta}$	e
sym2a	$e = \xi_\nu$ with $\nu = 0$
sym2b	$e = \eta$
sym2c	$e = \xi_\nu$ with $\nu = 1$
alt2a	$e = \xi_\nu$ with $\nu = 2$
alt2b	$e = \xi_\nu$ with $\nu = \frac{1}{2}$
alt2c	$e = \xi_\nu$ with $\nu = -1$
z1	$e = \xi_\nu$ with $\nu = e^{i\pi/3}$
z2	$e = \xi_\nu$ with $\nu = e^{-i\pi/3}$

We see that no of the considered tensors U admits a commutation symmetry $\text{sym}3$, $\text{sym}6$ or $\text{alt}6$. Furthermore the mentioned above connection between commutation symmetries and the maximal reduction of the length of $\mathfrak{P}_{i_1 \dots i_5}^{\text{red}}$ becomes visible. If $\dim V \geq 3$, then we obtain statement 4 of Theorem 1.11 from Theorem 5.5 and Tables 3 and 4.

6. TENSORS U GENERATED BY COVARIANT DERIVATIVES

In [15] we showed that examples of tensors which lie in an irreducible symmetry class belonging to $\lambda = (2\ 1) \vdash 3$ can be constructed from covariant derivatives of symmetric or alternating tensor fields of order 2. Such tensors are special examples of tensors U considered in the present paper. More precisely we proved the following facts in [15]:

Let M be a differentiable¹² *manifold* of dimension $\dim M \geq 1$ and ∇ be a *torsion-free covariant derivative* on M . Further let $\psi \in \mathcal{T}_2 M$, $\omega \in \mathcal{T}_2 M$ be covariant, differentiable tensor fields of order 2 which are *symmetric* or *skew-symmetric*, respectively. Then the tensors

$$(6.1) \quad (\nabla\psi - \text{sym}(\nabla\psi))|_p \quad , \quad (\nabla\omega - \text{alt}(\nabla\omega))|_p \quad , \quad p \in M$$

lie in irreducible symmetry classes¹³ belonging to $\lambda = (2\ 1) \vdash 3$. The operators 'sym' and 'alt' denote the *symmetrization* and *anti-symmetrization*, respectively. The right ideals that define the irreducible $(2\ 1)$ -symmetry classes for the tensors (6.1) are generated by the idempotents

$$(6.2) \quad h_s := e_s - f_s \quad , \quad h_a := e_a - f_a$$

where

$$(6.3) \quad e_s := \frac{1}{2} ([1, 2, 3] + [2, 1, 3]) \quad , \quad e_a := \frac{1}{2} ([1, 2, 3] - [2, 1, 3])$$

$$(6.4) \quad f_s := \frac{1}{6} \sum_{p \in \mathcal{S}_3} p \quad , \quad f_a := \frac{1}{6} \sum_{p \in \mathcal{S}_3} \text{sign}(p) p.$$

Note that this statement is based on the convention

$$(6.5) \quad (\nabla T)_{i_1 i_2 i_3} = T_{i_1 i_2 ; i_3} = \nabla_{i_3} T_{i_1 i_2}$$

for the numbering of the tensor indices.

A second result of [15] was that the minimal right ideals $h_s \cdot \mathbb{K}[\mathcal{S}_3]$ and $h_a \cdot \mathbb{K}[\mathcal{S}_3]$ are different from the critical right ideals $\mathfrak{r}_0 = f \cdot \mathbb{K}[\mathcal{S}_3]$ from Theorem 1.10. Thus the tensors (6.1) can be used in generator formulas for algebraic covariant derivative curvature tensors.

Now we want to clear the question which of the idempotents from

$$\{\eta\} \cup \{\xi_\nu \mid \nu \in \mathbb{K}\}$$

¹²Here the word "differentiable" denotes the class C^∞ .

¹³The symmetry classes of these tensor fields are different.

generate the right ideals $h_s \cdot \mathbb{K}[\mathcal{S}_3]$ and $h_a \cdot \mathbb{K}[\mathcal{S}_3]$. A short computer calculation [7, derivs.nb] by means of PERMS [10] yield

Theorem 6.1. *The idempotents h_s , h_a , η , ξ_ν satisfy the following relations:*

$$(6.6) \quad \eta \cdot h_s \neq h_s \quad , \quad \eta \cdot h_a \neq h_a$$

$$(6.7) \quad \xi_\nu \cdot h_s = h_s \quad \Leftrightarrow \quad \nu = 0$$

$$(6.8) \quad \xi_\nu \cdot h_a = h_a \quad \Leftrightarrow \quad \nu = 2.$$

Thus we obtain

$$(6.9) \quad \xi_0 \cdot \mathbb{K}[\mathcal{S}_3] = h_s \cdot \mathbb{K}[\mathcal{S}_3] \quad , \quad \xi_2 \cdot \mathbb{K}[\mathcal{S}_3] = h_a \cdot \mathbb{K}[\mathcal{S}_3].$$

It is interesting that $\nu = 0$ and $\nu = 2$ are ν -values which allow the construction of shortest formulas for $\mathfrak{P}_{i_1 \dots i_5}^{\text{red}}$ (see Tables 3 and 4). Note, however, that only $\nu = 2$ lead to tensors U which can be used to construct such shortest formulas both for symmetric and for skew-symmetric W . A tensor U with $\nu = 0$ will not produce a minimal length of $\mathfrak{P}_{i_1 \dots i_5}^{\text{red}}$ if W is an alternating tensor. If we now assume that $\dim M \geq 3$, then the equation system (3.7) yields a complete set of linear identities for the tensors (6.1) (see explanation of (4.1)) and we obtain Theorem 1.12.

APPENDIX A: FIRST REDUCTION OF $\mathfrak{P}_{i_1 \dots i_5}$

In Appendix A we present tensor coordinates of the tensors (1.19) which we calculated by means of the **Mathematica** packages **PERMS** [10] and **Ricci** [27]. **PERMS** contains a tool **Symmetrize** which makes it possible for us to apply symmetry operators $a \in \mathbb{K}[\mathcal{S}_r]$ of **Perms** onto the coordinates $T_{i_1 \dots i_r}$ of tensors $T \in \mathcal{T}_r V$ defined in **Ricci** and to form

$$(aT)_{i_1 \dots i_r} = \sum_{p \in \mathcal{S}_r} a(p) T_{i_{p(1)} \dots i_{p(r)}} .$$

We use the common symbol W_{ij} for the coordinates A_{ij} and S_{ij} of the tensors $A, S \in \mathcal{T}_2 V$. Then we apply the idempotent $\frac{1}{24} y_{t'}^* \in \mathbb{K}[\mathcal{S}_5]$ to the coordinates $U_{ijk} W_{lr}$ or $W_{ij} U_{klr}$ of the tensors $U \otimes W$ or $W \otimes U$, respectively. The resulting expressions

$$\begin{aligned} \frac{1}{24} (y_{t'}^*(U \otimes W))_{ijklr} &= \frac{1}{24} \{U_{ijk} W_{lr} - U_{ijl} W_{kr} - U_{ijr} W_{kl} + U_{ijr} W_{lk} \pm 44 \text{ terms}\} \\ \frac{1}{24} (y_{t'}^*(W \otimes U))_{ijklr} &= \frac{1}{24} \{W_{ij} U_{klr} - W_{ij} U_{lkr} - W_{ij} U_{rkl} + W_{ij} U_{rlk} \pm 44 \text{ terms}\} \end{aligned}$$

are automatically reduced by **Ricci** by means of the identity

$$W_{ij} = \epsilon W_{ji} \quad , \quad \epsilon \in \{1, -1\} .$$

We could get a further reduction of these coordinate expressions if we considered all linear identities which are fulfilled by the tensors of the symmetry class of U . However, we want to carry out such a reduction only in the following appendices. During the calculations of Appendix A we assume that U is a tensor "without any symmetry", i.e. $U \in \mathcal{T}_3 V$ is a tensor whose symmetry class is defined by the right ideal $\mathfrak{r} = \mathbb{K}[\mathcal{S}_3]$.

Under these assumptions we obtain the following two coordinate expressions for $\frac{1}{24} (y_{t'}^*(U \otimes W))_{ijklr}$ and $\frac{1}{24} (y_{t'}^*(W \otimes U))_{ijklr}$ (see [7, part1.nb]):

The coordinates $\frac{1}{24}(y_{t'}^*(U \otimes W))_{ijklr}$.

$$\begin{aligned}
& \frac{-1+\epsilon}{24} U_{klr} W_{ij} + \frac{1-\epsilon}{24} U_{lkr} W_{ij} + \frac{1-\epsilon}{24} U_{rkl} W_{ij} + \frac{-1+\epsilon}{24} U_{rlk} W_{ij} - \\
& \frac{1}{24} U_{jlr} W_{ik} - \frac{\epsilon}{24} U_{ljr} W_{ik} - \frac{\epsilon}{24} U_{rjl} W_{ik} - \frac{1}{24} U_{rlj} W_{ik} + \\
& \frac{1}{24} U_{jkr} W_{il} + \frac{\epsilon}{24} U_{kjr} W_{il} + \frac{\epsilon}{24} U_{rjk} W_{il} + \frac{1}{24} U_{rkj} W_{il} + \\
& \frac{1}{24} U_{jkl} W_{ir} - \frac{1}{24} U_{jlk} W_{ir} - \frac{1}{24} U_{klj} W_{ir} + \frac{1}{24} U_{lkj} W_{ir} + \\
& \frac{1}{24} U_{ilr} W_{jk} + \frac{\epsilon}{24} U_{lir} W_{jk} + \frac{\epsilon}{24} U_{ril} W_{jk} + \frac{1}{24} U_{rli} W_{jk} - \\
& \frac{1}{24} U_{ikr} W_{jl} - \frac{\epsilon}{24} U_{kir} W_{jl} - \frac{\epsilon}{24} U_{rik} W_{jl} - \frac{1}{24} U_{rki} W_{jl} - \\
& \frac{1}{24} U_{ikl} W_{jr} + \frac{1}{24} U_{ilk} W_{jr} + \frac{1}{24} U_{kli} W_{jr} - \frac{1}{24} U_{lki} W_{jr} + \\
& \frac{-1+\epsilon}{24} U_{ijr} W_{kl} + \frac{1-\epsilon}{24} U_{jir} W_{kl} + \frac{1-\epsilon}{24} U_{rij} W_{kl} + \frac{-1+\epsilon}{24} U_{rji} W_{kl} - \\
& \frac{1}{24} U_{ijl} W_{kr} + \frac{1}{24} U_{jil} W_{kr} + \frac{1}{24} U_{lij} W_{kr} - \frac{1}{24} U_{lji} W_{kr} + \\
& \frac{1}{24} U_{ijk} W_{lr} - \frac{1}{24} U_{jik} W_{lr} - \frac{1}{24} U_{kij} W_{lr} + \frac{1}{24} U_{kji} W_{lr}
\end{aligned}$$

The coordinates $\frac{1}{24}(y_{t'}^*(W \otimes U))_{ijklr}$.

$$\begin{aligned}
& \frac{1-\epsilon}{24} U_{klr} W_{ij} + \frac{-1+\epsilon}{24} U_{lkr} W_{ij} + \frac{-1+\epsilon}{24} U_{rkl} W_{ij} + \frac{1-\epsilon}{24} U_{rlk} W_{ij} - \\
& \frac{\epsilon}{24} U_{jlr} W_{ik} - \frac{1}{24} U_{ljr} W_{ik} - \frac{1}{24} U_{rjl} W_{ik} - \frac{\epsilon}{24} U_{rlj} W_{ik} + \\
& \frac{\epsilon}{24} U_{jkr} W_{il} + \frac{1}{24} U_{kjr} W_{il} + \frac{1}{24} U_{rjk} W_{il} + \frac{\epsilon}{24} U_{rkj} W_{il} + \\
& \frac{\epsilon}{24} U_{jkl} W_{ir} - \frac{\epsilon}{24} U_{jlk} W_{ir} - \frac{\epsilon}{24} U_{klj} W_{ir} + \frac{\epsilon}{24} U_{lkj} W_{ir} + \\
& \frac{\epsilon}{24} U_{ilr} W_{jk} + \frac{1}{24} U_{lir} W_{jk} + \frac{1}{24} U_{ril} W_{jk} + \frac{\epsilon}{24} U_{rli} W_{jk} - \\
& \frac{\epsilon}{24} U_{ikr} W_{jl} - \frac{1}{24} U_{kir} W_{jl} - \frac{1}{24} U_{rik} W_{jl} - \frac{\epsilon}{24} U_{rki} W_{jl} - \\
& \frac{\epsilon}{24} U_{ikl} W_{jr} + \frac{\epsilon}{24} U_{ilk} W_{jr} + \frac{\epsilon}{24} U_{kli} W_{jr} - \frac{\epsilon}{24} U_{lki} W_{jr} + \\
& \frac{1-\epsilon}{24} U_{ijr} W_{kl} + \frac{-1+\epsilon}{24} U_{jir} W_{kl} + \frac{-1+\epsilon}{24} U_{rij} W_{kl} + \frac{1-\epsilon}{24} U_{rji} W_{kl} - \\
& \frac{\epsilon}{24} U_{ijl} W_{kr} + \frac{\epsilon}{24} U_{jil} W_{kr} + \frac{\epsilon}{24} U_{lij} W_{kr} - \frac{\epsilon}{24} U_{lji} W_{kr} + \\
& \frac{\epsilon}{24} U_{ijk} W_{lr} - \frac{\epsilon}{24} U_{jik} W_{lr} - \frac{\epsilon}{24} U_{kij} W_{lr} + \frac{\epsilon}{24} U_{kji} W_{lr}
\end{aligned}$$

Furthermore our calculations in the notebook [7, part1.nb] yield the following

Lemma: *The above tensor coordinates satisfy*

$$\frac{1}{24}(y_{t'}^*(U \otimes W))_{ijklr} = \epsilon \frac{1}{24}(y_{t'}^*(W \otimes U))_{ijklr}$$

if $W_{ji} = \epsilon W_{ij}$, $\epsilon \in \{1, -1\}$. The expressions possesses 32 summands if W is symmetric and 40 summands if W is an alternating tensor.

APPENDIX B: SHORTEST $\mathfrak{P}_{i_1 \dots i_5}^{\text{red}}$ WITH U BELONGING TO η

In Appendix B we present formulas of minimal length for $\frac{1}{24}(y_{t'}^*(U \otimes W))_{ijklr}$ where U is a tensor from the symmetry class defined by the idempotent η from (2.22). Such formulas of minimal length arise if $\mathcal{P} = \{[1, 2, 3], [1, 3, 2]\}$. Linear identities which characterize the symmetry of U in this case are given in (4.7).

We obtain a length of 12 summands if W is symmetric ($\epsilon = 1$) and a length of 20 summands if W is an alternating tensor ($\epsilon = -1$). The determination of the formulas is described in Section 4.1. The computer calculations for Appendix B can be found in the **Mathematica** notebook [7, part12b.nb].

The coordinates $\frac{1}{24}(y_{t'}^*(U \otimes W))_{ijklr}$ for $\mathcal{P} = \{[1, 2, 3], [1, 3, 2]\}$, $\epsilon = 1$.

$$\begin{aligned} & \frac{1}{12} U_{jrl} W_{ik} - \frac{1}{12} U_{jrk} W_{il} + \frac{1}{12} U_{jkl} W_{ir} - \frac{1}{12} U_{jlk} W_{ir} - \\ & \frac{1}{12} U_{irl} W_{jk} + \frac{1}{12} U_{irk} W_{jl} - \frac{1}{12} U_{ikl} W_{jr} + \frac{1}{12} U_{ilk} W_{jr} - \\ & \frac{1}{6} U_{ijl} W_{kr} - \frac{1}{12} U_{ilj} W_{kr} + \frac{1}{6} U_{ijk} W_{lr} + \frac{1}{12} U_{ikj} W_{lr} \end{aligned}$$

The coordinates $\frac{1}{24}(y_{t'}^*(U \otimes W))_{ijklr}$ for $\mathcal{P} = \{[1, 2, 3], [1, 3, 2]\}$, $\epsilon = -1$.

$$\begin{aligned} & -\frac{1}{3} U_{klr} W_{ij} - \frac{1}{6} U_{krl} W_{ij} - \frac{1}{6} U_{jlr} W_{ik} - \frac{1}{12} U_{jrl} W_{ik} + \\ & \frac{1}{6} U_{jkr} W_{il} + \frac{1}{12} U_{jrk} W_{il} + \frac{1}{12} U_{jkl} W_{ir} - \frac{1}{12} U_{jlk} W_{ir} + \\ & \frac{1}{6} U_{ilr} W_{jk} + \frac{1}{12} U_{irl} W_{jk} - \frac{1}{6} U_{ikr} W_{jl} - \frac{1}{12} U_{irk} W_{jl} - \\ & \frac{1}{12} U_{ikl} W_{jr} + \frac{1}{12} U_{ilk} W_{jr} - \frac{1}{3} U_{ijr} W_{kl} - \frac{1}{6} U_{irj} W_{kl} - \\ & \frac{1}{6} U_{ijl} W_{kr} - \frac{1}{12} U_{ilj} W_{kr} + \frac{1}{6} U_{ijk} W_{lr} + \frac{1}{12} U_{ikj} W_{lr} \end{aligned}$$

APPENDIX C: GENERIC CASE FOR $\mathfrak{P}_{i_1 \dots i_5}^{\text{red}}$ WITH U BELONGING TO ξ_ν

In Appendix C we present formulas of minimal length for $\frac{1}{24}(y_\nu^*(U \otimes W))_{ijklr}$ where U is a tensor from the symmetry class defined by the idempotent ξ_ν from (2.21).

First we consider the example of the set $\mathcal{P} = \{[1, 2, 3], [1, 3, 2]\}$ and restrict us to such $\nu \in \mathbb{K}$ for which $\Delta_{\mathcal{P}}(\nu) \neq 0$. Linear identities which characterize the symmetry of U in this case are given in (4.13).

We obtain a length of 16 summands if W is symmetric ($\epsilon = 1$) and a length of 20 summands if W is an alternating tensor ($\epsilon = -1$). The determination of the formulas is described in Section 4.2. The computer calculations can be found in the **Mathematica** notebook [7, part12a.nb].

$\mathfrak{P}_{i_1 \dots i_5}^{\text{red}}$ for $\mathcal{P} = \{[1, 2, 3], [1, 3, 2]\}$.

The coordinates $\frac{1}{24}(y_\nu^*(U \otimes W))_{ijklr}$ for $\epsilon = 1$.

$$\begin{aligned}
& \frac{-(-1+2\nu)}{24(-1+\nu)(1+\nu)} U_{jlr} W_{ik} + \frac{\nu(-1+2\nu)}{24(-1+\nu)(1+\nu)} U_{jrl} W_{ik} + \\
& \frac{-1+2\nu}{24(-1+\nu)(1+\nu)} U_{jkr} W_{il} - \frac{\nu(-1+2\nu)}{24(-1+\nu)(1+\nu)} U_{jrk} W_{il} + \\
& \frac{-1+2\nu}{24(-1+\nu)} U_{jkl} W_{ir} - \frac{-1+2\nu}{24(-1+\nu)} U_{jlk} W_{ir} + \\
& \frac{-1+2\nu}{24(-1+\nu)(1+\nu)} U_{ilr} W_{jk} - \frac{\nu(-1+2\nu)}{24(-1+\nu)(1+\nu)} U_{irl} W_{jk} - \\
& \frac{-1+2\nu}{24(-1+\nu)(1+\nu)} U_{ikr} W_{jl} + \frac{\nu(-1+2\nu)}{24(-1+\nu)(1+\nu)} U_{irk} W_{jl} - \\
& \frac{-1+2\nu}{24(-1+\nu)} U_{ikl} W_{jr} + \frac{-1+2\nu}{24(-1+\nu)} U_{ilk} W_{jr} - \\
& \frac{(-1+2\nu)^2}{24(-1+\nu)(1+\nu)} U_{ijl} W_{kr} - \frac{(-2+\nu)(-1+2\nu)}{24(-1+\nu)(1+\nu)} U_{ilj} W_{kr} + \\
& \frac{(-1+2\nu)^2}{24(-1+\nu)(1+\nu)} U_{ijk} W_{lr} + \frac{(-2+\nu)(-1+2\nu)}{24(-1+\nu)(1+\nu)} U_{ikj} W_{lr}
\end{aligned}$$

The coordinates $\frac{1}{24}(y_{t'}^*(U \otimes W))_{ijklr}$ for $\epsilon = -1$.

$$\begin{aligned}
& \frac{-(-1+2\nu)^2}{12(-1+\nu)(1+\nu)} U_{klr} W_{ij} - \frac{(-2+\nu)(-1+2\nu)}{12(-1+\nu)(1+\nu)} U_{krl} W_{ij} - \\
& \frac{(-1+2\nu)^2}{24(-1+\nu)(1+\nu)} U_{jlr} W_{ik} - \frac{(-2+\nu)(-1+2\nu)}{24(-1+\nu)(1+\nu)} U_{jrl} W_{ik} + \\
& \frac{(-1+2\nu)^2}{24(-1+\nu)(1+\nu)} U_{jkr} W_{il} + \frac{(-2+\nu)(-1+2\nu)}{24(-1+\nu)(1+\nu)} U_{jrk} W_{il} + \\
& \frac{-1+2\nu}{24(-1+\nu)} U_{jkl} W_{ir} - \frac{-1+2\nu}{24(-1+\nu)} U_{jlk} W_{ir} + \\
& \frac{(-1+2\nu)^2}{24(-1+\nu)(1+\nu)} U_{ilr} W_{jk} + \frac{(-2+\nu)(-1+2\nu)}{24(-1+\nu)(1+\nu)} U_{irl} W_{jk} - \\
& \frac{(-1+2\nu)^2}{24(-1+\nu)(1+\nu)} U_{ikr} W_{jl} - \frac{(-2+\nu)(-1+2\nu)}{24(-1+\nu)(1+\nu)} U_{irk} W_{jl} - \\
& \frac{-1+2\nu}{24(-1+\nu)} U_{ikl} W_{jr} + \frac{-1+2\nu}{24(-1+\nu)} U_{ilk} W_{jr} - \\
& \frac{(-1+2\nu)^2}{12(-1+\nu)(1+\nu)} U_{ijr} W_{kl} - \frac{(-2+\nu)(-1+2\nu)}{12(-1+\nu)(1+\nu)} U_{irj} W_{kl} - \\
& \frac{(-1+2\nu)^2}{24(-1+\nu)(1+\nu)} U_{ijl} W_{kr} - \frac{(-2+\nu)(-1+2\nu)}{24(-1+\nu)(1+\nu)} U_{ilj} W_{kr} + \\
& \frac{(-1+2\nu)^2}{24(-1+\nu)(1+\nu)} U_{ijk} W_{lr} + \frac{(-2+\nu)(-1+2\nu)}{24(-1+\nu)(1+\nu)} U_{ikj} W_{lr}
\end{aligned}$$

Remark: We see that no denominator of the coefficients in the above expressions has the root $\nu = \frac{1}{2}$ both for $\epsilon = 1$ and for $\epsilon = -1$. Furthermore every numerator of the above coefficients contains a factor $(-1+2\nu)$. Consequently we obtain $\frac{1}{24}(y_{t'}^*(U \otimes W))_{ijklr}|_{\nu=1/2} = 0$ both for $\epsilon = 1$ and for $\epsilon = -1$. This illustrates that a tensor U from the symmetry class of $\xi_{1/2}$ can not be used to generate a non-trivial algebraic covariant derivative curvature tensor.

Now we repeat the above considerations for the set $\mathcal{P} = \{[1, 2, 3], [3, 2, 1]\}$. For this set \mathcal{P} the polynomial $\Delta_{\mathcal{P}}(\nu)$ possesses the only root $\nu = \frac{1}{2}$, which is the critical ν -value for our construction of algebraic covariant derivative curvature tensors from the tensors U and W .

Since no other ν -values has to be excluded in this case, the formulas for $\mathfrak{P}_{i_1 \dots i_5}^{\text{red}}$ are formulas which yield algebraic covariant derivative curvature tensors for every $\nu \neq \frac{1}{2}$ (see Remark 4.2). The computer calculations can be found in the **Mathematica** notebook [7, part16a.nb].

For $\mathcal{P} = \{[1, 2, 3], [3, 2, 1]\}$ Procedure 3.6 yields the following linear identities (3.13) for U :

$$\begin{array}{rclcl}
- \frac{\nu^2 - \nu + 1}{2\nu - 1} U_{ijk} & + & \frac{\nu^2 - 1}{2\nu - 1} U_{kji} & + & U_{ikj} & = & 0 \\
\frac{\nu^2 - \nu + 1}{2\nu - 1} U_{ijk} & - & \frac{\nu^2 - 2\nu}{2\nu - 1} U_{kji} & + & U_{jik} & = & 0 \\
- \frac{\nu^2 - 2\nu}{2\nu - 1} U_{ijk} & + & \frac{\nu^2 - \nu + 1}{2\nu - 1} U_{kji} & + & U_{jki} & = & 0 \\
\frac{\nu^2 - 1}{2\nu - 1} U_{ijk} & - & \frac{\nu^2 - \nu + 1}{2\nu - 1} U_{kji} & + & U_{kij} & = & 0
\end{array}$$

If we carry out the Procedure 3.8 by means of these identities, we obtain the following expressions for $\mathfrak{P}_{i_1 \dots i_5}^{\text{red}}$:

$\mathfrak{P}_{i_1 \dots i_5}^{\text{red}}$ for $\mathcal{P} = \{[1, 2, 3], [3, 2, 1]\}$.

The coordinates $\frac{1}{24}(y_{t'}^*(U \otimes W))_{ijklr}$ for $\epsilon = 1$.

$$\begin{aligned}
& \frac{-1 + \nu}{24} U_{jlr} W_{ik} - \frac{\nu}{24} U_{rlj} W_{ik} + \frac{1 - \nu}{24} U_{jkr} W_{il} + \frac{\nu}{24} U_{rkj} W_{il} + \\
& \frac{2 - \nu}{24} U_{jkl} W_{ir} + \frac{1 + \nu}{24} U_{lkj} W_{ir} + \frac{1 - \nu}{24} U_{ilr} W_{jk} + \frac{\nu}{24} U_{rli} W_{jk} + \\
& \frac{-1 + \nu}{24} U_{ikr} W_{jl} - \frac{\nu}{24} U_{rki} W_{jl} + \frac{-2 + \nu}{24} U_{ikl} W_{jr} + \frac{-1 - \nu}{24} U_{lki} W_{jr} + \\
& \frac{-1 - \nu}{24} U_{ijl} W_{kr} + \frac{-2 + \nu}{24} U_{lji} W_{kr} + \frac{1 + \nu}{24} U_{ijk} W_{lr} + \frac{2 - \nu}{24} U_{kji} W_{lr}
\end{aligned}$$

The coordinates $\frac{1}{24}(y_{t'}^*(U \otimes W))_{ijklr}$ for $\epsilon = -1$.

$$\begin{aligned}
& \frac{-1 - \nu}{12} U_{klr} W_{ij} + \frac{-2 + \nu}{12} U_{rlk} W_{ij} + \frac{-1 - \nu}{24} U_{jlr} W_{ik} + \frac{-2 + \nu}{24} U_{rlj} W_{ik} + \\
& \frac{1 + \nu}{24} U_{jkr} W_{il} + \frac{2 - \nu}{24} U_{rkj} W_{il} + \frac{2 - \nu}{24} U_{jkl} W_{ir} + \frac{1 + \nu}{24} U_{lkj} W_{ir} + \\
& \frac{1 + \nu}{24} U_{ilr} W_{jk} + \frac{2 - \nu}{24} U_{rli} W_{jk} + \frac{-1 - \nu}{24} U_{ikr} W_{jl} + \frac{-2 + \nu}{24} U_{rki} W_{jl} + \\
& \frac{-2 + \nu}{24} U_{ikl} W_{jr} + \frac{-1 - \nu}{24} U_{lki} W_{jr} + \frac{-1 - \nu}{12} U_{ijr} W_{kl} + \frac{-2 + \nu}{12} U_{rji} W_{kl} + \\
& \frac{-1 - \nu}{24} U_{ijl} W_{kr} + \frac{-2 + \nu}{24} U_{lji} W_{kr} + \frac{1 + \nu}{24} U_{ijk} W_{lr} + \frac{2 - \nu}{24} U_{kji} W_{lr}
\end{aligned}$$

Remark: During the above calculation the critical factor $(-1 + 2\nu)$ was canceled in all fractions within the expressions for $\frac{1}{24}(y_{t'}^*(U \otimes W))_{ijklr}$. Thus we could set $\nu = \frac{1}{2}$ in these formulas. However it is clear that the resulting formulas do not describe an algebraic covariant derivative curvature tensor, because the assumption $\nu \neq \frac{1}{2}$ was the foundation of our calculation. The above formulas represent an algebraic covariant derivative curvature tensor only then if the coordinates of U also satisfy the above linear identities of type (3.13). However, these identities are not defined for $\nu = \frac{1}{2}$.

APPENDIX D: NON-GENERIC CASES FOR $\mathfrak{P}_{i_1 \dots i_5}^{\text{red}}$ WITH U BELONGING TO ξ_ν

Now we present examples for a further reduction of the length of the formulas for $\frac{1}{24}(y_{t'}^*(U \otimes W))_{ijklr}$ from Appendix C. Again we assume that U is a tensor from the symmetry class defined by the idempotent ξ_ν from (2.21). However we use such a value ν which is a root of a polynomial $P_q^{\mathcal{P}}(\nu)$ occuring in the considered expression from Appendix C. The meaning of $P_q^{\mathcal{P}}(\nu)$ is defined in (4.11). The vanishing of $P_q^{\mathcal{P}}(\nu)$ leads to a reduction of the length of the considered expression of type (4.11).

Our consideration is based on the set $\mathcal{P} = \{[1, 2, 3], [3, 2, 1]\}$. For this set the values $\nu = -1$ and $\nu = 2$ lead to the minimal length of $\mathfrak{P}_{i_1 \dots i_5}^{\text{red}}$ both for symmetric W ($\epsilon = 1$) and for alternating W ($\epsilon = -1$). We obtain a length of 12 summands if W is symmetric ($\epsilon = 1$) and a length of 10 summands if W is an alternating tensor ($\epsilon = -1$). The determination of these results is described in Section 4.2.2. The computer calculations can be found in the **Mathematica** notebook [7, roots16a.nb].

The below expressions are coordinates of algebraic covariant derivative curvature tensors only if the coordinates of U satisfy the correct linear identities describing the symmetry of U . These identities can be obtained if one sets $\nu = -1$ or $\nu = 2$ into the general identities for U given in the second part of Appendix C.

$\mathfrak{P}_{i_1 \dots i_5}^{\text{red}}$ containing a U from the symmetry class defined by ξ_{-1} .

The coordinates $\frac{1}{24}(y_{t'}^*(U \otimes W))_{ijklr}$ for $\epsilon = 1$.

$$\begin{aligned} & -\frac{1}{12}U_{jlr}W_{ik} + \frac{1}{24}U_{rlj}W_{ik} + \frac{1}{12}U_{jkr}W_{il} - \frac{1}{24}U_{rkj}W_{il} + \frac{1}{8}U_{jkl}W_{ir} + \\ & \frac{1}{12}U_{ilr}W_{jk} - \frac{1}{24}U_{rli}W_{jk} - \frac{1}{12}U_{ikr}W_{jl} + \frac{1}{24}U_{rki}W_{jl} - \frac{1}{8}U_{ikl}W_{jr} - \\ & \frac{1}{8}U_{lji}W_{kr} + \frac{1}{8}U_{kji}W_{lr} \end{aligned}$$

The coordinates $\frac{1}{24}(y_{t'}^*(U \otimes W))_{ijklr}$ for $\epsilon = -1$.

$$\begin{aligned} & -\frac{1}{4}U_{rlk}W_{ij} - \frac{1}{8}U_{rlj}W_{ik} + \frac{1}{8}U_{rkj}W_{il} + \frac{1}{8}U_{jkl}W_{ir} + \frac{1}{8}U_{rli}W_{jk} - \\ & \frac{1}{8}U_{rki}W_{jl} - \frac{1}{8}U_{ikl}W_{jr} - \frac{1}{4}U_{rji}W_{kl} - \frac{1}{8}U_{lji}W_{kr} + \frac{1}{8}U_{kji}W_{lr} \end{aligned}$$

$\mathfrak{P}_{i_1 \dots i_5}^{\text{red}}$ containing a U from the symmetry class defined by ξ_2 .

The coordinates $\frac{1}{24}(y_{t'}^*(U \otimes W))_{ijklr}$ for $\epsilon = 1$.

$$\begin{aligned} & \frac{1}{24}U_{jlr}W_{ik} - \frac{1}{12}U_{rlj}W_{ik} - \frac{1}{24}U_{jkr}W_{il} + \frac{1}{12}U_{rkj}W_{il} + \frac{1}{8}U_{lkj}W_{ir} - \\ & \frac{1}{24}U_{ilr}W_{jk} + \frac{1}{12}U_{rli}W_{jk} + \frac{1}{24}U_{ikr}W_{jl} - \frac{1}{12}U_{rki}W_{jl} - \frac{1}{8}U_{lki}W_{jr} - \\ & \frac{1}{8}U_{ijl}W_{kr} + \frac{1}{8}U_{ijk}W_{lr} \end{aligned}$$

The coordinates $\frac{1}{24}(y_{t'}^*(U \otimes W))_{ijklr}$ for $\epsilon = -1$.

$$\begin{aligned} & -\frac{1}{4}U_{klr}W_{ij} - \frac{1}{8}U_{jlr}W_{ik} + \frac{1}{8}U_{jkr}W_{il} + \frac{1}{8}U_{lkj}W_{ir} + \frac{1}{8}U_{ilr}W_{jk} - \\ & \frac{1}{8}U_{ikr}W_{jl} - \frac{1}{8}U_{lki}W_{jr} - \frac{1}{4}U_{ijr}W_{kl} - \frac{1}{8}U_{ijl}W_{kr} + \frac{1}{8}U_{ijk}W_{lr} \end{aligned}$$

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REFERENCES

- [1] N. Blažić, N. Bokan and P. Gilkey. A note on Osserman Lorentzian manifolds. *Bull. London Math. Soc.*, 29: 227–230, 1997.
- [2] H. Boerner. *Darstellungen von Gruppen*, volume 74 of *Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen*. Springer-Verlag, Berlin, Göttingen, Heidelberg, 1955.
- [3] H. Boerner. *Representations of Groups*. North-Holland Publishing Company, Amsterdam, 2, revised edition, 1970.
- [4] Q.-S. Chi. A curvature characterization of certain locally rank-one symmetric spaces. *J. Differential Geom.*, 28: 187–202, 1988.
- [5] M. Clausen and U. Baum. *Fast Fourier Transforms*. BI Wissenschaftsverlag, Mannheim, Leipzig, Wien, Zürich, 1993.
- [6] M. Clausen and U. Baum. Fast Fourier transforms for symmetric groups. In L. Finkelstein and W. M. Kantor, editors, *Groups and Computation*, volume 11 of *DIMACS Series in Discrete Mathematics and Theoretical Computer Science*, pages 27 – 39, Providence, Rhode Island, 1993. DIMACS, American Mathematical Society. Proc. DIMACS workshop Rutgers University 1991.
- [7] B. Fiedler. Examples of calculations by means of PERMS. Mathematica notebooks. Internet <http://home.t-online.de/home/Bernd.Fiedler.RoschStr.Leipzig/pnbks.htm>.
- [8] B. Fiedler. A characterization of the dependence of the Riemannian metric on the curvature tensor by Young symmetrizers. *Z. Anal. Anw.*, 17(1): 135 – 157, 1998.
- [9] B. Fiedler. *An Algorithm for the Decomposition of Ideals of Semi-Simple Rings and its Application to Symbolic Tensor Calculations by Computer*. Habilitationsschrift, Universität Leipzig, Leipzig, Germany, November 1999. Fakultät für Mathematik und Informatik.
- [10] B. Fiedler. *PERMS 2.1 (15.1.1999)*. Mathematisches Institut, Universität Leipzig, Leipzig, 1999. Will be sent in to MathSource, Wolfram Research Inc.
- [11] B. Fiedler. Characterization of tensor symmetries by group ring subspaces and computation of normal forms of tensor coordinates. In A. Betten, A. Kohnert, R. Laue and A. Wassermann, editors, *Algebraic Combinatorics and Applications. Proceedings of the Euroconference, ALCOMA, Gößweinstein, Germany, September 12–19, 1999*, pages 118–133, Berlin, 2001. Springer-Verlag.
- [12] B. Fiedler. Ideal decompositions and computation of tensor normal forms. In *Séminaire Lotharingien de Combinatoire*, 2001. El. published: <http://www.mat.univie.ac.at/~slc>. B45g, 16 pp. Archive: <http://arXiv.org/abs/math.CO/0211156>.
- [13] B. Fiedler. Determination of the structure of algebraic curvature tensors by means of Young symmetrizers. In *Séminaire Lotharingien de Combinatoire*, 2002. Electronically published: <http://www.mat.univie.ac.at/~slc>. B48d, 20 pp. Preprint: <http://arXiv.org/abs/math.CO/0212278>.

- [14] B. Fiedler. Generators of algebraic covariant derivative curvature tensors and Young symmetrizers. Preprint: <http://arXiv.org/abs/math.CO/0310020>, 2003. 18 pages. Chapter for a book "Progress in Computer Science Research", in preparation by Nova Science Publishers, Inc.
- [15] B. Fiedler. On the symmetry classes of the first covariant derivatives of tensor fields. In *Séminaire Lotharingien de Combinatoire*, 2003. Accepted by the electronic journal. 21 pp. Preprint: <http://arXiv.org/abs/math.CO/0301042>.
- [16] S. A. Fulling, R. C. King, B. G. Wybourne and C. J. Cummins. Normal forms for tensor polynomials: I. The Riemann tensor. *Class. Quantum Grav.*, 9: 1151 – 1197, 1992.
- [17] W. Fulton and J. Harris. *Representation Theory: A First Course*, volume 129 of *Graduate Texts in Mathematics, Readings in Mathematics*. Springer-Verlag, New York, Berlin, Heidelberg, London, Paris, Tokyo, Hong Kong, Barcelona, Budapest, 1991.
- [18] William Fulton. *Young Tableaux*. Number 35 in London Mathematical Society Student Texts. Cambridge University Press, Cambridge, New York, Melbourne, 1997.
- [19] E. García-Riό, D. N. Kupeli and M. E. Vázquez-Abal. On a problem of Osserman in Lorentzian geometry. *Differential Geom. Appl.*, 7: 85–100, 1997.
- [20] P. B. Gilkey. Geometric properties of the curvature operator. In W. H. Chen, A.-M. Li, U. Simon, L. Verstraelen, C. P. Wang and M. Wiehe, editors, *Proceedings of Beijing Conference "Geometry and Topology of Submanifolds"*, volume 10, pages 62–70, Singapore, 2000. World Scientific.
- [21] P. B. Gilkey. *Geometric Properties of Natural Operators Defined by the Riemann Curvature Tensor*. World Scientific Publishing Co., Singapore, New Jersey, London, Hong Kong, 2001.
- [22] G. D. James and A. Kerber. *The Representation Theory of the Symmetric Group*, volume 16 of *Encyclopedia of Mathematics and its Applications*. Addison-Wesley Publishing Company, Reading, Mass., London, Amsterdam, Don Mills, Ont., Sidney, Tokyo, 1981.
- [23] A. Kerber. *Representations of Permutation Groups*, volume 240, 495 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, Heidelberg, New York, 1971, 1975.
- [24] A. Kerber. *Algebraic combinatorics via finite group actions*. BI-Wiss.-Verl., Mannheim, Wien, Zürich, 1991.
- [25] A. Kerber and A. Kohnert. *SYMMETRICA 1.0*. Lehrstuhl II für Mathematik, Department of Mathematics, University of Bayreuth, Bayreuth, 1994. Manual.
- [26] A. Kerber, A. Kohnert and A. Lascoux. SYMMETRICA, an object oriented computer-algebra system for the symmetric group. *J. Symbolic Computation*, 14: 195 – 203, 1992.
- [27] J. M. Lee, D. Lear, J. Roth, J. Coskey and L. Nave. *Ricci. A Mathematica package for doing tensor calculations in differential geometry. User's Manual. Version 1.32*. Department of Mathematics, Box 354350, University of Washington, Seattle, WA 98195-4350, 1992 - 1998. Ricci's home page: <http://www.math.washington.edu/~lee/Ricci/>.
- [28] D. E. Littlewood. *The Theory of Group Characters and Matrix Representations of Groups*. Clarendon Press, Oxford, 2. edition, 1950.
- [29] I. G. Macdonald. *Symmetric Functions and Hall Polynomials*. Clarendon Press, Oxford, 1979.
- [30] W. Müller. *Darstellungstheorie von endlichen Gruppen*. Teubner Studienbücher Mathematik. B. G. Teubner, Stuttgart, 1980.
- [31] M. A. Naimark and A. I. Štern. *Theory of Group Representations*, volume 246 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, Heidelberg, New York, 1982.
- [32] Y. Nikolayevsky. Osserman conjecture in dimension $n \neq 8, 16$. preprint, 2002. <http://arXiv.org/abs/math.DG/0204258>.
- [33] R. Osserman. Curvature in the eighties. *Amer. Math. Monthly*, 97: 731–756, 1990.
- [34] B. L. van der Waerden. *Algebra*, volume I, II. Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, Hong Kong, Barcelona, Budapest, 9., 6. edition, 1993.

- [35] H. Weyl. *The Classical Groups, their Invariants and Representations*. Princeton University Press, Princeton, New Jersey, 1939.

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